Homework 4

Due: Friday, September 15

1. Let S be a graded ring, and let I and J be homogeneous ideals in S.[1] Prove that each of the following is a homogeneous ideal .
   (a) $\sqrt{I}$.
   (b) $I + J$.
   (c) IJ.
   (d) $I \cap J$.

   You may, of course, assume that each of these is actually an ideal...

2. Let S be a graded ring, and let $I \subset S$ be a homogeneous ideal. Prove that I is prime if and only if for every pair of *homogeneous* $f, g \in S$ with $fg \in I$, one of f and g is in I.

3. Suppose that $f, g \in k[X_0, \cdots, X_n]$ are both homogeneous of degree d. Show that the function $P \mapsto f(P)/g(P)$ is well-defined on $\mathbb{P}^n$, outside the vanishing locus of g.

[1] In class on Wednesday, we will make the following definitions: A graded ring is a ring S equipped with a decomposition of abelian groups

$$S = \oplus_{d \geq 0} S_d$$

such that $S_d \cdot S_e \subseteq S_{d+e}$. Elements of S_d are called homogeneous of degree d. The existence of the decomposition means that if $f \in S$, then f can be written uniquely as $f = \sum_{d \geq 0} f_d$, where $f_d \in S_d$ and $f_d = 0$ for all but finitely many d.

An ideal $I \subset S$ is called homogeneous if

$$I = \oplus (I \cap S_d).$$

If this is true, then $f = \sum f_d$ is in I if and only if each $f_d \in I$.

We will show:

Lemma If S is a graded ring, and if $I \subset S$ is an ideal, then I is homogeneous if and only if it can be generated by homogeneous elements.