Homework 3

Due: *Monday* September 11

1. Show that the image of a morphism need not be closed. (HINT: *Remember, any regular function on X defines a morphism $X \to \mathbb{A}^1$.*)

2. Let $X \subset \mathbb{A}^n$ be an irreducible variety.

   (a) Show that the coordinate ring $k[X]$ is an integral domain.
   Therefore, we may construct the field of rational functions on X, $k(X) := \mathrm{Frac}\, k[X]$. Any element of $k(X)$ is represented by some $p(X)/q(X)$, where $p, q \in k[X]$ and $q \not\equiv 0$.

   (b) Let $f \in k(X)$ be a rational function. If $P \in X$, f is called *regular at P* if there is a representative $f = p/q$ where $q(P) \neq 0$. Let

   $$\mathrm{dom}(f) = \{P \in X : f \text{ is regular at } P\}.$$

   Prove that $\mathrm{dom}(f)$ is an open subset of X.

3. (a) Let $W \subset \mathbb{A}^n$ be closed, and suppose that $U \subset W$. Show that $\mathcal{I}(U) = \mathcal{I}(W)$ if and only if U is dense in W.

   (b) Let $\phi : V \to W$ be a morphism. Show that the image of ϕ is dense in W if and only if $\phi^* : k[W] \to k[V]$ is injective. *Such a morphism is called* dominant.

4. (a) Let

   $$\Delta_{\mathbb{A}^n} = \{(P, P) : P \in \mathbb{A}^n\} \subset \mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}.$$

   Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be coordinates on $\mathbb{A}^n \times \mathbb{A}^n \cong \mathbb{A}^{2n}$. What is $\mathcal{I}(\Delta_{\mathbb{A}^n})$? Show that $\Delta_{\mathbb{A}^n}$ is closed.

   (b) Suppose $V \subset \mathbb{A}^n$ is closed. Show that

   $$\Delta_V := \{(P, P); P \in V\} \subset \mathbb{A}^n \times \mathbb{A}^n$$

   is closed. (HINT: *$V \times V \subset \mathbb{A}^{2n}$ is closed.*)

5. Let $\phi : V \to W$ be a morphism.

   (a) The graph of ϕ is

   $$\Gamma_\phi := \{(P, \phi(P)) : P \in V\} \subset V \times W.$$

   Show that Γ_ϕ is closed. (HINT: *Consider the inverse image of Δ_W under $(\phi \times \mathrm{id}_W) : V \times W \to W \times W$.*)

   (b) Let $\psi : V \to W$ be a morphism. Show that $\{x \in V : \psi(x) = \psi(x)\}$ is closed.