Homework 2

Due: Friday, September 1

1. Let k be an algebraically closed field, and suppose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$. Show that there is no common solution $f_1 = f_2 = \cdots = 0$ if and only if there are $a_1, \cdots, a_r \in k[x_1, \cdots, x_n]$ such that
$$\sum_{i=1}^{r} a_i f_i = 1.$$

2. Let $\phi : R \to S$ be a ring homomorphism, and let $J \subset S$ be an ideal. Let $I = \phi^{-1}(J)$.
   (a) Show that I is an ideal of R.
   (b) Show that if J is prime, then I is prime.
   (c) Give an example to show that even if J is maximal, I need not be maximal.

   Extra credit: Suppose that $R = k[x_1, \cdots, x_m]/\mathfrak{a}$ and $S = k[y_1, \cdots, y_n]/\mathfrak{b}$, where k is algebraically closed. Show that if $J \subset S$ is maximal, then $\phi^{-1}(J)$ is maximal.

3. There is a natural identification (of sets) $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^2$. Show that the Zariski topology on $\mathbb{A}^2$ is strictly finer than the product topology of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

   Concretely, show:
   (a) Suppose $C_1, \cdots, C_r$ and $D_1, \cdots, D_r$ are closed subsets of $\mathbb{A}^1$. Then
   $$\cup_{i=1}^{r} C_i \times D_i \subset \mathbb{A}^2 \tag{1}$$
   is closed.
   (b) Find a set $S \subset \mathbb{A}^2$ which is closed but is *not* of the form (1).

4. If $f \in k[x_1, \cdots, x_n]$, the associated distinguished[1] affine open set is
$$D(f) := \{P \in \mathbb{A}^n : f(P) \neq 0\}.$$
   (a) Suppose $f, g \in k[x_1, \cdots, x_n]$. Show that $D(fg) = D(f) \cap D(g)$.
   (b) Show that the collection of distinguished open sets in $\mathbb{A}^n$ is a basis for the Zariski topology on $\mathbb{A}^n$.
   Recall that if X is a topological space, then a collection of open subsets $\mathcal{C}$ is a basis for the topology on X if for every open set U of X, and each $x \in U$, there is some $V \in \mathcal{C}$ such that
   $$x \in V \subseteq U.$$

5. Prove that $\mathbb{A}^n$ is compact.

[1] or principal, or standard, or basic

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