1. Let $X \subset \mathbb{P}^n$ be a collection of $r$ distinct points. Prove directly that for $\ell \gg 0$, $h_X(\ell) = r$.

2. The dual space of $\mathbb{P}^2$ is $\mathbb{P}^{2*}$, the space of lines in $\mathbb{P}^2$. In fact, $\mathbb{P}^{2*}$ is isomorphic to $\mathbb{P}^2$; a point $[a_0, a_1, a_2] \in \mathbb{P}^{2*}$ corresponds to $Z_P(a_0X_0 + a_1X_1 + a_2X_2)$. (Note that is well-defined on equivalence classes!)

Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous form, and let $X = Z(F)$ be the associated plane curve, with smooth locus $X^{sm}$.

Show that the map

\[ X^{sm} \xrightarrow{\phi} \mathbb{P}^{2*} \]

\[ P \longleftarrow T_P X \]

(where $T_P X$ is the closure of the external tangent space to $X$ at $P$) is a morphism, by giving an explicit formula for $\phi$ in terms of $F$ and the coordinates on $\mathbb{P}^2$.

The closure of the image is called the dual curve $X^*$.

3. \*Continue to assume $X = Z(F) \subset \mathbb{P}^2$.

(a) Show that the set of $L \in \mathbb{P}^{2*}$ which pass through a singular point of $X$ is a proper, closed subset of $\mathbb{P}^{2*}$.

(b) Show that the set of $L \in \mathbb{P}^{2*}$ which are tangent to $X$ is a proper, closed subset of $\mathbb{P}^{2*}$.

(c) Suppose $\deg F = d$. Show that there is an open subset $U \subset \mathbb{P}^{2*}$ such that for each $L \in U$, $L \cap X$ consists of exactly $d$ points.

4. Let $Y \subset \mathbb{P}^n$ be a closed subset of dimension $r$, with Hilbert polynomial $P_Y$. The arithmetic genus of $Y$ is

\[ p_a(Y) = (-1)^r (P_Y(0) - 1). \]

(a) Show that $p_a(\mathbb{P}^n) = 0$.

(b) Suppose $Y$ is a curve in $\mathbb{P}^2$ of degree $d$. Show that $p_a(Y) = \frac{(d-1)(d-2)}{2}$.

(c) Suppose $Y$ is a hypersurface of degree $d$ in $\mathbb{P}^n$. What is the arithmetic genus of $Y$?

5. \*Suppose $X \subset \mathbb{P}^n$ is closed. Show that $X$ is linear if and only if its degree is one.