Homework 13

Due: Friday, December **8**

1. Let $X \subset \mathbb{P}^n$ be a collection of r distinct points. Prove directly that for $\ell \gg 0$, $h_X(\ell) = r$.

2. The *dual space* of $\mathbb{P}^2$ is $\mathbb{P}^{2*}$, the space of lines in $\mathbb{P}^2$. In fact, $\mathbb{P}^{2*}$ is isomorphic to $\mathbb{P}^2$; a point $[a_0, a_1, a_2] \in \mathbb{P}^{2*}$ corresponds $\mathcal{Z}_{\mathbb{P}}(a_0X_0 + a_1X_1 + a_2X_2)$. (Note that is well-defined on equivalence classes!)

   Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous form, and let $X = \mathcal{Z}(F)$ be the associated plane curve, with smooth locus X^{sm}.

   Show that the map

   $$X^{\mathrm{sm}} \xrightarrow{\phi} \mathbb{P}^{2*}$$

   $$P \longmapsto T_P X$$

   (where $T_P X$ is the closure of the *external* tangent space to X at P) is a morphism, by giving an explicit formula for ϕ in terms of F and the coordinates on $\mathbb{P}^2$.

   The closure of the image is called the dual curve X^*.

3. [*] Continue to assume $X = \mathcal{Z}(F) \subset \mathbb{P}^2$.

   (a) Show that the set of $L \in \mathbb{P}^{2*}$ which pass through a singular point of X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (b) Show that the set of $L \in \mathbb{P}^{2*}$ which are tangent to X is a proper, closed subset of $\mathbb{P}^{2*}$.

   (c) Suppose $\deg F = d$. Show that there is an open subset $U \subset \mathbb{P}^{2*}$ such that for each $L \in U$, $L \cap X$ consists of exactly d points.

4. Let $Y \subset \mathbb{P}^n$ be a closed subset of dimension r, with Hilbert polynomial P_Y. The *arithmetic genus* of Y is

   $$p_a(Y) = (-1)^r (P_Y(0) - 1).$$

   (a) Show that $p_a(\mathbb{P}^n) = 0$.

   (b) Suppose Y is a curve in $\mathbb{P}^2$ of degree d. Show that $p_a(Y) = \frac{(d-1)(d-2)}{2}$.

   (c) Suppose Y is a hypersurface of degree d in $\mathbb{P}^n$. What is the arithmetic genus of Y?

5. [*] Suppose $X \subset \mathbb{P}^n$ is closed. Show that X is linear if and only if its degree is one.