Homework 12

Due: Friday, ??

1. The ring of *dual numbers* is $k[\epsilon] = k[t]/(t^2)$, where ϵ is the coset $t + (t^2)$. Note that as a vector space, $k[\epsilon] = \{a_0 + a_1\epsilon : a_0, a_1 \in k\}$. *Make sure you understand how multiplication works in this ring.*

   Let X be a variety, and let $P \in X$.

   (a) Let $\alpha : \mathcal{O}_{P,X} \to k[\epsilon]$ be a ring homomorphism. Define a map $D_\alpha : \mathcal{O}_{P,X} \to k$ as follows: For $f \in \mathcal{O}_{P,X}$, write $\alpha(f) = a_0 + a_1\epsilon$, and let $D_\alpha(f) = a_1$. Show that D_α is a k-linear derivation.

   (b) Conversely, given $D \in \mathrm{Der}_k(\mathcal{O}_{P,X}, k)$, explain how to find $\alpha \in \mathrm{Hom}(\mathcal{O}_{P,X}, k[\epsilon])$ such that $D = D_\alpha$.

2. Fix $n \in \mathbb{N}$, and consider $\mathrm{Mat}_n(k[\epsilon])$. Note that any $M \in \mathrm{Mat}_N(k[\epsilon])$ can be written as $M_0 + \epsilon M_1$, where $M_0, M_1 \in \mathrm{Mat}_n(k)$. Let I_n be the identity matrix.

   (a) Suppose $A, B \in \mathrm{Mat}_n(k[\epsilon])$. Show that
   $$(I_n + \epsilon A)(I_n + \epsilon B) = I_n + \epsilon(A + B).$$

   (b) Show that for any $A \in \mathrm{Mat}_n(k)$, $I_n + \epsilon A \in \mathrm{GL}_n(k[\epsilon])$.

   (c) Describe those $A \in \mathrm{Mat}_2(k)$ for which $I_2 + \epsilon A \in \mathrm{SL}_2(k[\epsilon])$.

   Extra credit: Do the last question for general n.

3. Prove that a polynomial $f \in k[T]$ is a local parameter at the point $T = \alpha$ if and only if α is a simple root of f.

The next questions are combinatorial, not algebro-geometric; but we'll be using the results later on.

4. A *numerical polynomial* is a polynomial $p(z) \in \mathbb{Q}[z]$ such that $p(n) \in \mathbb{Z}$ for all sufficiently large integers. For example, the binomial coefficient function
   $$\binom{z}{r} := \frac{z(z-1)\cdots(z-r+1)}{r!}$$
   is a numerical polynomial. (Set $\binom{z}{0} = 1$.)

   For any function f on the integers, define its *difference function* to be $\Delta(f) : n \mapsto f(n+1) - f(n)$. *You should read, but not hand in, (a), (b) and (c).*

   (a) The sum and difference of numerical polynomials are numerical polynomials. $\Delta(f - g) = \Delta(f) - \Delta(g)$.

(b) If f is a numerical polynomial of degree $r \geq 1$, then $\Delta(f)$ is a numerical polynomial of degree $r-1$.

(c) Suppose that $p(z) \in \mathbb{Q}[z]$ (not necessarily numerical) has degree r. Show that there are unique numbers $c_0, \cdots, c_r \in \mathbb{Q}$ so that

$$p(z) = \sum_{j=0}^{r} c_j \binom{z}{j}. \tag{1}$$

We will call this the *binomial representation* of $p(z)$.

(d) Express $\Delta(\binom{z}{r})$ as a binomial coefficient function.

(e) Given a binomial representation of $p(z)$ as in (1), find a binomial representation for $\Delta(p)$.

(f) Suppose $p(z)$ is a numerical polynomial, that $c_0 \in \mathbb{Q}$ and that $p(z) - c_0$ is also a numerical polynomial. Then $c_0 \in \mathbb{Z}$.

5. Suppose p is a numerical polynomial. Prove that in the binomial representation (1), each $c_j \in \mathbb{Z}$. (HINT: *Prove by induction on* $\deg p$*, using the* Δ *operator.*)

We will use these to show the following:

> *Let $f : \mathbb{Z} \to \mathbb{Z}$ be any function. Suppose that there exists a numerical polynomial q such that $\Delta(f)(n) = q(n)$ for all $n \gg 0$. Then there exists a numerical polynomial p such that $f(n) = p(n)$ for all $n \gg 0$.*