Homework 11

Due: Friday, November 10

1. Consider the Steiner surface $S \subset \mathbb{A}^3$ defined as the zero locus of

$$F(x, y, z) = x^2y^2 - x^2z^2 + y^2z^2 - xyz.$$

   (a) Find (explicit, simple) equations for T, the singular locus of S.

   (b) Find (explicit, simple) equations for W, the singular locus of T.

   This is an example of the stratification we spoke about in class.

2. Let X be an affine variety. Fix natural numbers m and n, and let $A = (a_{ij}) \in \mathrm{Mat}_{m\times n}(k[X])$ be an $m \times n$ matrix of functions on A. For each nonnegative integer r, define

$$X_{A,\le r} = \{P \in X : \mathrm{rk}(a_{ij}(P)) \le r\}$$
$$X_{A,r} = \{P \in X : \mathrm{rk}(a_{ij}(P)) = r\}$$

   (a) Show that $X_{A,\le r}$ is a closed subset of X.

   (b) What can you say about $X_{A,=r}$?

3. [*] Let X be variety. Show that X can be written as a finite, disjoint union of smooth, locally closed subsets.

4. [*] Let $f(x, y) \in k[x, y]$ be a nonconstant polynomial. Let $X = \mathcal{Z}(f)$, and let $P = (a, b) \in X$. Assign weight 1 to $x - a$ and $y - b$. Then (HW9#4) f has a Taylor expansion centered at P:

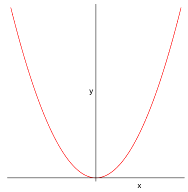
$$\begin{aligned} f &= f(P) + (\frac{\partial}{\partial x} f(P)(x-a) + \frac{\partial}{\partial y} f(P)(y-b)) + \\ &\quad \frac{1}{2}(\frac{\partial^2}{\partial x^2} f(P)(x-a)^2 + 2\frac{\partial^2}{\partial x \partial y} f(P)(x-a)(y-b) + \frac{\partial^2}{\partial y^2} f(P)(y-b)^2) + \cdots \\ &= \sum f_i \end{aligned}$$

   where f_i is homogeneous (in this new weighting) of degree i.

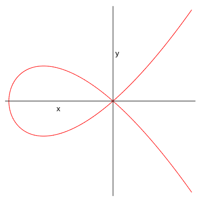
   The *initial term*, or leading term, of f at P, $\mathrm{In}_P(f)$, is the nonzero term f_i of smallest degree. The *tangent cone* of X at P is $\mathcal{Z}(\mathrm{In}_P(f))$.

   For each of the following planar curves C, compute the tangent cone of C at $P = (0, 0)$. Graph the real points of this tangent cone.

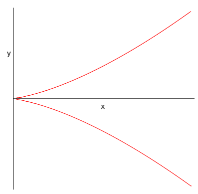
(a) $\alpha(x, y) = y - x^2$.



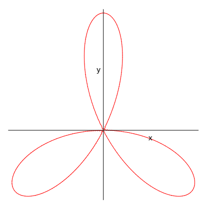
(b) $\beta(x, y) = y^2 - x^3 - x^2$



(c) $\gamma(x, y) = y^2 - x^3$



(d) $\delta(x, y) = (x^2 + y^2)^2 + 3x^2y - y^3$



5. Suppose $F \in k[x, y]$ is homogeneous of degree $d \geq 1$. Show that there is a factorization

$$F(x, y) = x^{e_0} \prod_{j=1}^{r} (y - a_i x)^{e_i}$$

where $a_i \neq a_j$ for $i \neq j$.

If $F = \mathrm{In}_{(0,0)}(f)$, then $\deg F$ is the multiplicity of $\mathcal{Z}(f)$ at the origin, and e_i is the multiplicity of $\mathcal{Z}(f)$ along the line $\mathcal{Z}(y - a_i x)$.