Homework 10

Due: Friday, November 3

1. $\boxed{*}$Consider the curve $Y = \mathcal{Z}(xy - x^4 - y^4) \subset \mathbb{A}^2$. Recall that the blowup of $\mathbb{A}^2$ at $(0,0)$ is $\mathcal{Z}(xV - yU) \subset \mathbb{A}^2 \times \mathbb{P}^1$, where $[U,V]$ are coordinates on $\mathbb{P}^1$. Let π be the projection $\pi : \mathcal{Z}(xV - yU) \to \mathbb{A}^2$.
   - What is $\pi^{-1}(Y - \{(0,0)\}) \subset \mathbb{A}^2 \times \mathbb{P}^1$?
   - What is $\pi^{-1}((0,0))$?
   - Let $\widetilde{Y}$ be the closure of $\pi^{-1}(Y - \{0\})$ in $\mathbb{A}^2 \times \mathbb{P}^1$. What is $\widetilde{Y} \cap \pi^{-1}((0,0))$?

2. Let $R = k[x,y]/(xy - x^4 - y^4)$, and let $K = \operatorname{Frac} R$.
   (a) Show that R is not integrally closed in K.
   (b) What is the integral closure of R in K?

3. $\boxed{*}$*Cremona, revisited* Consider the rational map

$$\mathbb{P}^2 \overset{\phi}{\dashrightarrow} \mathbb{P}^2$$

$$[a_0, a_1, a_2] \longmapsto [a_1 a_2, a_0 a_2, a_0 a_1]$$

   defined on $\mathbb{P}^2 - \{[1,0,0],[0,1,0],[0,0,1]\}$. Let X_0, X_1 and X_2 be coordinates on the first copy of $\mathbb{P}^2$, and let Y_0, Y_1 and Y_2 be coordinates on the second copy.

   The closure in $\mathbb{P}^2 \times \mathbb{P}^2$ of the graph of ϕ is the correspondence $Z \subset \mathbb{P}^2 \times \mathbb{P}^2$ given by

$$Z = \mathcal{Z}_{\mathbb{P}}(X_0 Y_0 - X_1 Y_1, X_1 Y_1 - X_2 Y_2).$$

   Let $P = [a_0, a_1, a_2]$.
   (a) Suppose $a_0 a_1 a_2 \neq 0$. What is $Z[P]$?
   (b) Suppose $a_0 = 0$, $a_1 a_2 \neq 0$. What is $Z[P]$?
   (c) Suppose $a_0 = a_1 = 0$, $a_2 \neq 0$. What is $Z[P]$?

4. Let W, X and Y be irreducible projective varieties. Suppose $Z_{WX} \subset W \times X$ and $Z_{XY} \subset X \times Y$ are correspondences. Explain how to "compose" these to obtain a correspondence $Z \subset W \times Y$. If $P \in W$, what is $Z[P]$?