Homework 1

Due: Friday, August 25

1. Let $S \subset \mathbb{A}^n$ be any subset. Prove that
$$\mathcal{I}(S) := \{f \in k[x_1, \cdots, x_n] : \forall P \in S, f(P) = 0\}$$
is an ideal of $k[x_1, \cdots, x_n]$.

2. Describe all maximal ideals of each of the following rings.
   (a) $\mathbb{C}[x]$
   (b) $\mathbb{R}[x]$
   (c) $\mathbb{Q}[x]$

3. (a) Define affine sets by
$$X = \{(a_1, a_2) : a_2^2 = a_1^3 - a_1\} \subset \mathbb{A}^2$$
$$Y = \{(b_1, b_2, b_3) : b_2^2 + b_1 = b_3 b_1 \text{ and } b_1^2 = b_3\} \subset \mathbb{A}^3$$
Consider the map
$$\mathbb{A}^2 \xrightarrow{\alpha} \mathbb{A}^3$$
$$(a_1, a_2) \longmapsto (a_1, a_2, a_1^2)$$
Show that $\alpha(X) \subseteq Y$.

   (b) Define ideals by
$$I = (x_2^2 - x_1^3 + x_1) \subset k[x_1, x_2]$$
$$J = (t_2^2 + t_1 - t_3 t_1, t_1^2 - t_3) \subset k[t_1, t_2, t_3]$$
Consider the map
$$k[t_1, t_2, t_3] \xrightarrow{\beta} k[x_1, x_2]$$
$$t_1 \longmapsto x_1$$
$$t_2 \longmapsto x_2$$
$$t_3 \longmapsto x_1^2$$
Show that $\beta(J) \subseteq I$.

4. Consider the following ideals in $k[x, y, z]$:

$$I_1 = (xy + y^2, xz + yz)$$
$$I_2 = (xy + y^2, xz + yz + xyz + y^2 z)$$
$$I_3 = (xy^2 + y^3, xz + yz)$$

For which $i, j \in \{1, 2, 3\}$ do we have

(a) $I_i = I_j$?

(b) $\mathcal{Z}(I_i) = \mathcal{Z}(I_j)$?

(HINT: *Factor!*)