12 Hilbert polynomials

12.1 Calibration

Let $X \subset \mathbb{P}^n$ be a (not necessarily irreducible) closed algebraic subset. In this section, we'll look at a device which measures the way X sits inside $\mathbb{P}^n$.

Throughout this section, let $S = k[X_0, \cdots, X_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$, and let $S(X)$ be the homogeneous coordinate ring of X. Then $S(X)$ is a graded module over the graded ring S. Define the *Hilbert function* of X by

$$h_X(\ell) = \dim S(X)_\ell,$$

the dimension of the ℓ^{th} graded piece of the coordinate ring of X. Recall that this is isomorphic to $S_\ell / \mathcal{I}(X)_\ell$.

Example $h_{\mathbb{P}^n}(\ell) = \binom{n+\ell}{\ell}$. We saw this before when were examining the Veronese embedding.

This means that

$$h_X(\ell) = h_{\mathbb{P}^n}(\ell) - \dim_k \mathcal{I}(X)_\ell$$

Example Suppose that X consists of three distinct points in $\mathbb{P}^2$. They either do or don't live on a line...

- If the points are collinear, then there is a linear relation in $\mathcal{I}(X)$, and $h_X(1) = 2 - 1 = 1$.
- If the points are not collinear, then there is not a linear relation in $\mathcal{I}(X)$, and $h_X(1) = 2 - 0 = 2$.

Having said that, we have:

Claim If X consists of three distinct points in $\mathbb{P}^2$, then $h_X(2) = 3$.

Proof For each point P_i, let L_i be a linear form which vanishes on P_i, but not on the others. Then the product $L_i L_j$ is a quadratic function which vanishes on P_i and P_j, but not the other. This gives a surjective map from $S(\mathbb{P}^2)_2$ to the space of functions on X, so that $h_X(3) = \binom{2+2}{2} - 3 = 3$. $\square$

In fact, for all $\ell \geq 3$, $h_X(\ell) = 3$, no matter what position the points are. Two things worth pointing out:

- For small ℓ, $h_X(\ell)$ depends on the arrangment of X.

- For sufficiently large ℓ, this difference is erased.

Here is another example. Suppose X is a hypersurface in $\mathbb{P}^n$; say $\mathcal{I}(X) = (F)$, with $F \subset k[X_0, \cdots, X_n]$ of degree d. Then $\mathcal{I}(X)_m$ is the polynomials of degree m which are divisible by F. This means that "multiplication by F" gives an isomorphism between S_{m-d} and $\mathcal{I}(X)_m$, so that

$$h_X(m) = h_{\mathbb{P}^n}(m) - h_{\mathbb{P}^n}(m-d)$$

Assume $\ell \geq d$; then

$$= \binom{n+\ell}{\ell} - \binom{n+\ell-d}{\ell-d}$$

For instance, if $n = 2$, then

$$= d\ell - \frac{d(d-3)}{2}.$$

There's a regularity there which is independent of the curve. The goal of this chapter is to generalize these remarks...

12.2 Algebra

12.2.1 Numerical polynomials

See homework. The point is that a function $h : \mathbb{N} \to \mathbb{Z}$ is called a numerical polynomial if there's some $P \in \mathbb{Q}[z]$ such that, for $\ell \gg 0$, $h(\ell) = P(\ell)$.

12.2.2 Hilbert polynomials of graded modules

Let S be a graded noetherian ring. A S-module M is *graded* if it comes equipped with a decomposition

$$M = \oplus M_d$$

such that $S_d M_e \subseteq M_{d+e}$.

If $\ell \in \mathbb{N}$, the twist of M by ℓ is the same module with the grading shifted:

$$M(\ell)_d = M_{d+\ell}.$$

The annihilator of M is

$$\mathrm{Ann}(M) = \{x \in S : x \cdot M = 0\}.$$

If M is graded, this is a homogeneous ideal in S.

Lemma Let M be a finitely generated graded module over S. There exists a filtration

$$0 \subseteq M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$$

by graded submodules such that for each i,

$$M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(\ell_i)$$

where $\mathfrak{p}_i \subset S$ is a homogeneous prime ideal and $\ell \in \mathbb{Z}$.

Proof The proof is just like the existence of prime ideal factorizations in noetherian rings.

Given any M, let $\mathcal{F}(M)$ be the set of graded submodules of M which admit such a filtration. It's nonempty, since (0) is in this class. Let $M' \subset M$ be a maximal element; it exists, since M is a noetherian module. Consider $M'' = M/M'$. If $M'' = 0$, we're done.

Otherwise, consider the collection of ideals which are annihilators of homogeneous elements,

$$\mathcal{I} = \{I_m = \operatorname{Ann}(m) : m \in M'' - \{0\} \text{ homogeneous }\}.$$

Then each I_m is a proper homogeneous ideal. By noetherianness, we can take a maximal element I_m of $\mathcal{I}$.

Claim I_m is prime.

Suppose $a, b \in S$, $ab \in I_m$, $b \notin I_m$; we'll show $a \in I_m$. By taking homogeneous components, we may assume a and b are homogeneous. Consider $bm \in M''$. Since $b \notin I_m$, $bm \neq 0$. Since $I_m \subseteq bm$, by maximality $I_m = I_{bm}$; but then $abm = 0$, so $a \in I_{bm} = I_m$.

So, I_m is a homogeneous prime ideal, call it $\mathfrak{p}$. Let $\deg(m) = \ell$. Then the module $N \subset M''$ generated by m is isomorphic to $(S/\mathfrak{p})(-\ell)$. Lift this:

$$\begin{array}{ccc} N' & \longrightarrow & M \\ & & \\ N \cong (S/\mathfrak{p})(-\ell) & \longrightarrow & M'' \end{array}$$

Then $M' \subseteq N'$, $N'/M' \cong (S/\mathfrak{p})(-\ell)$; N' has a suitable filtration, which contradicts the maximality of M'. Therefore, $M' = M$. □

The prime ideals $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}$ which show up should be thought of as the "elementary divisors" of the S-module M. Among them, we distinguish the minimal ones; these are the minimal primes of M.

Lemma Let $\mathfrak{p} \subseteq S$ be a homogeneous prime ideal. Then $\mathfrak{p} | \operatorname{Ann}(M)$ if and only if $\mathfrak{p} \supseteq \mathfrak{p}_i$ for one of the minimal primes $\mathfrak{p}_i$ of M.

Proof $\mathfrak{p} \supseteq \mathrm{Ann}(M)$ if and only if $\mathfrak{p}$ annihilates some M^i/M^{i-1}. But $\mathrm{Ann}((S/\mathfrak{p}_i)(\ell)) = \mathfrak{p}_i$. □

Recall that the localization of S at a prime ideal $\mathfrak{p}$ is

$$S_{\mathfrak{p}} = \{\frac{x}{s} : \deg x = \deg s, s \notin \mathfrak{p}\}.$$

Lemma Let $\mathfrak{p}$ be a minimal prime of M, and choose any filtration as above. Then the number of i for which $\mathfrak{p}_i = \mathfrak{p}$ is the length of $M_{\mathfrak{p}}$ as $S_{\mathfrak{p}}$-module, where $S_{\mathfrak{p}}$ is the localization.

Proof Choose some filtration. For each i with $\mathfrak{p}_i \neq \mathfrak{p}$, there exists $a \in \mathfrak{p}_i$ which is a unit in $S_{\mathfrak{p}}$, so that

$$\begin{aligned} M^i_{\mathfrak{p}}/M^{i-1}_{\mathfrak{p}} &\cong S_{\mathfrak{p}} \otimes_S (S/\mathfrak{p}_i) \\ &\cong (0). \end{aligned}$$

(Use the fact that $1 \otimes 1 = (a^{-1} \cdot a) \otimes 1 = a^{-1} \otimes a = 0$.)

On the other hand, if $\mathfrak{p} = \mathfrak{p}_i$, then

$$\begin{aligned} M^i_{\mathfrak{p}}/M^{i-1}_{\mathfrak{p}} &\cong S_{\mathfrak{p}} \otimes (S/\mathfrak{p}) \\ &\cong (S/\mathfrak{p}). \end{aligned}$$

Therefore, $M_{\mathfrak{p}}$ is an S_p-module of the advertised length. □

Definition The multiplicity of M at $\mathfrak{p}$ is $\mu_{\mathfrak{p}}(M)$, the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$.

12.3 Hilbert-Serre theorem

Attached to a graded module is the Hilbert function,

$$h_M(\ell) = \dim_k M_\ell.$$

Theorem [Hilbert-Serre] Let $S = k[x_0, \cdots, x_n]$ with the standard grading, and let M be a finitely generated graded S-module. Then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that, for $\ell \gg 0$, $h_M(\ell) = P_M(\ell)$. Moreover, $\deg P_M = \dim \mathcal{Z}_{\mathbb{P}^n}(\mathrm{Ann}(M))$.

Remark We assign $\deg 0 = -1$, and $\dim \emptyset = -1$.

Proof A homomorphism of graded modules necessarily preserves the grading. Therefore, any short exact sequence of graded modules

$$0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow 0$$

yields the equality

$$h_{M^1}(\ell) + h_{M^2}(\ell) = h_{M^3}(\ell)$$

for each ℓ. After choosing a filtration as above, it suffices to prove the theorem for $(S/\mathfrak{p})(\ell)$. Since $h_{M(\ell)}(d) = h_M(d+\ell)$, it suffices to prove the theorem for $M = (S/\mathfrak{p})$. We distinguish two cases, the first of which is the base case and the second is an inductive step, by induction on $\dim \mathcal{Z}(\mathrm{Ann}(M))$.

- Suppose $\mathfrak{p} = (X_0, \cdots, X_n)$. Then $h_M(\ell) = 0$ for $\ell > 0$, and thus for $\ell \gg 0$. Moreover, $\mathcal{Z}(\mathfrak{p}) = \emptyset$, so that $\dim \mathcal{Z}(\mathfrak{p}) = \deg h_M = -1$.
- Otherwise, suppose $X_i \notin \mathfrak{p}$. Then – remember that $\deg x_i = 1$ – we have an exact sequence

$$0 \longrightarrow M(-1) \xrightarrow{x_i} M \longrightarrow M'' = M/x_iM \longrightarrow 0,$$

  and $h_{M''}(\ell) = h_M(\ell) - h_M(\ell-1) = (\Delta h_M)(\ell-1)$.

  Now, $\mathcal{Z}(\mathrm{Ann}(M'')) = \mathcal{Z}(\mathfrak{p}) \cap \mathcal{Z}(x_i)$. The hyperplane section $\mathcal{Z}(x_i)$ doesn't contain $\mathcal{Z}(\mathfrak{p})$ (by hypothesis), so that $\dim \mathcal{Z}(\mathrm{Ann}(M'')) = \dim \mathcal{Z}(\mathfrak{p}) - 1$. By induction, $h_{M''}$ is a numerical polynomial, represented by a polynomial $P_{M''}$ of degree $\dim \mathcal{Z}(\mathrm{Ann}(M''))$. Since (Δh_M) is a numerical polynomial of degree $\dim \mathcal{Z}(\mathrm{Ann}(M)) - 1$, h_M is a numerical polynomial of degree $\dim \mathcal{Z}(\mathrm{Ann}(M))$.

□

The only missing ingredient is the lemma promised in the homework:

Lemma Let $f : \mathbb{Z} \to \mathbb{Z}$ be any function. Suppose that there exists a numerical polynomial q such that $\Delta(f)(n) = q(n)$ for all $n \gg 0$. Then there exists a numerical polynomial p such that $f(n) = p(n)$ for all $n \gg 0$.

Proof By the homework, there exists a polynomial $Q(z) = \sum_{j=0}^{r} c_j \binom{z}{j}$ such that, for $n \gg 0$, $Q(n) = q(n)$. Let

$$P(z) = \sum_{j=0}^{r} c_j \binom{z}{j+1}.$$

Then for $n \gg 0$, $\Delta(P) = \Delta(f) = Q(n) = q(n)$, so that $\Delta(P-f)(n) = 0$ for $n \gg 0$. Therefore, $(P - f)(n) = a$ for some $a \in \mathbb{Z}$ and all $n \gg 0$, so that $f(n) = P(n) - a$ is a numerical polynomial. □

12.4 Geometry of the Hilbert polynomial

Definition Suppose $Y \subset \mathbb{P}^n$. The Hilbert polynomial P_Y of Y is the Hilbert polynomial of its homogeneous coordinate ring $S(Y)$. Note that $\deg P_Y = \dim Y$. The *degree* of Y is $(\dim Y)!$ times the leading coefficient of P_Y.

Remark Since P_Y is a numerical polynomial, it has a binomial representation

$$P_Y(z) = \sum_{j=0}^{r} c_j \binom{z}{j}. \tag{1}$$

with $c_j \in \mathbb{Z}$; then $\deg Y = c_r$.

Lemma If $Y \subset \mathbb{P}^n$ is nonempty, then $\deg(Y) \in \mathbb{N}$.

Proof Since $Y \neq \emptyset$, P_Y is a nonzero polynomial of degree $r = \dim Y$. Then $\deg Y = c_r \in \mathbb{Z}$, as above. It's positive since there are functions on Y of arbtirarily large degree, so that $h_Y(\ell) > 0$ infinitely often. □

Lemma Suppose $Y \subset \mathbb{P}^n$, $Y = Y_1 \cup Y_2$, $\dim Y_1 = \dim Y_2 = r$, $\dim(Y_1 \cap Y_2) < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$.

Proof Let $I_j = \mathcal{I}(Y_j)$, and let $I = I_1 \cap I_2 = \mathcal{I}(Y)$. Note that $\mathcal{I}(Y_1 \cap Y_2) = \sqrt{(I_1 + I_2)}$. There is an exact sequence

$$0 \longrightarrow S/I \longrightarrow S/I_1 \oplus S/I_2 \longrightarrow S/(I_1 + I_2) \longrightarrow 0$$

Since $\dim(Y_1 \cap Y_2) < r$, $\deg P_{S/(I_1+I_2)} < r$, and the leading coefficient of S/I is the sum of the leading coefficients of those of S/I_1 and S/I_2. □

Lemma $\deg \mathbb{P}^n = 1$

Proof Use the earlier calculation of $P_{\mathbb{P}^n}(z)$. □

Lemma If $H \subset \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree d, then $\deg H = d$.

Proof Suppose $\mathcal{I}(H) = (F)$, F homogeneous of degree d. There's a diagram of graded modules

$$0 \longrightarrow S(-d) \xrightarrow{F} S \longrightarrow S(H) = S/(F) \longrightarrow 0$$

Then

$$\begin{aligned} h_H(\ell) &= h_{\mathbb{P}^n}(\ell) - h_{\mathbb{P}^n}(\ell - d) \\ h_H(z) &= h_{\mathbb{P}^n}(z) - h_{\mathbb{P}^n}(z - d) \\ &= \binom{z+n}{n} - \binom{z-d+n}{n} \\ &= \frac{d}{(n-1)!} z^{n-1} + \cdots \end{aligned}$$

and the degree of H is d. □

Remark The degree depends not just on the variety, but on the way it sits in the ambient projective space. *Do an example of the d-uple embedding?*

12.5 Intersection theory

Let $X \subset \mathbb{P}^n$ be a projective variety, not necessarily irreducible, of pure dimension r. Let $H \subset \mathbb{P}^n = \mathcal{Z}(F)$ be some hypersurface – choose F reduced. By the principal ideal theorem,

$$X \cap H = Z_1 \cup \cdots \cup Z_m,$$

where $\dim Z_j = r - 1$. Each Z_j corresponds to a homogeneous prime ideal $\mathfrak{p}_j = \mathcal{I}_X(Z_j)$.

Let $S = k[X_0, \cdots, X_n]$. Then we have homogeneous coordinate rings

$$\begin{aligned} S(X) &= S/\mathcal{I}(X) \\ S(H) &= S/\mathcal{I}(H) = S/(F) \\ S(X \cap H) &= S/\sqrt{(\mathcal{I}(X) + \mathcal{I}(H))} \end{aligned}$$

Note that, as S-module, $\mathrm{Ann}(S(X \cap H)) = \mathcal{I}(X) + \mathcal{I}(H)$, and the minimal primes of this module are simply the $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$.

Definition The intersection multiplicity of X and H along Z_j is

$$i(X, H; Z_j) = \mu_{p_j}(S/(\mathcal{I}(X) + \mathcal{I}(H))).$$

Example Consider the elliptic curve $E = \mathcal{Z}_{\mathbb{P}}(Y^2Z - X^3 + XZ^2) \subset \mathbb{P}^2$, and assume we're not in characteristic three. Let H be the hyperplane $H = \mathcal{Z}(Y)$. Then we compute intersection multiplicities by looking at the module

$$\begin{aligned} M &= \frac{k[X,Y,Z]}{(Y^2Z - X^3 + XZ^2, Y)} \\ &= \frac{k[X,Y,Z]}{(X^3 - XZ^2, Y)} \qquad \cong \frac{k[X,Z]}{(X^3 - XZ^2)} \\ &\cong \frac{k[X,Z]}{X(X-Z)(X+Z)} \end{aligned}$$

Note that as a k-module, this has dimension three. There are three maximal ideals of $k[X,Z]$ which contain the ideal $(X^3 - XZ^2)$, namely, (X), $(X - Z)$, $(X + Z)$. These are the three (minimal) prime ideals associated with this module. Let, say, $\mathfrak{p} = (X - Z)$. If we localize $k[X,Y]$ at $\mathfrak{p}$, then we invert (in particular) X and $X + Z$, so that

$$M_{\mathfrak{p}} \cong k[X,Z]/(X - Z)$$

has length one as $M_{\mathfrak{p}}$-module. The same is true for the other ideals.

Example Same elliptic curve, but now consider the hyperplane $W = \mathcal{Z}(X - Z)$. Then the module in question is

$$\begin{aligned} N &= \frac{k[X,Y,Z]}{(Y^2Z - X^3 + XZ^2, X - Z)} \\ &= \frac{k[X,Y]}{(Y^2X - X^3 + X^3)} \\ &= \frac{k[X,Y]}{(Y^2X)}. \end{aligned}$$

The minimal primes of this module are (Y) and (X), corresponding to the (projective) points $[1,0,1]$ and $[0,1,0]$ (the "point at infinity"), respectively. The multiplicities at these points are two and one, respectively.

Theorem With all notation as above (especially, $X \not\subset H$),

$$\sum_{j=1}^{s} i(X, H; Z_j) \cdot \deg(Z_j) = (\deg X)(\deg H).$$

Proof Suppose $\deg H = d$, $H = \mathcal{Z}(F)$, F reduced. Let $M = S/(\mathcal{I}(X) + \mathcal{I}(H))$. As before, we have an exact sequence

$$0 \longrightarrow S(X)(-d) \xrightarrow{F} S(X) \longrightarrow M \longrightarrow 0$$

so that

$$P_M(z) = P_X(z) - P_X(z-d).$$

Consider a (maximal) filtration $M^0 \subset M^1 \subset \cdots \subset M^q = M$, with quotients $M^i/M^{i-1} \cong (S/\mathfrak{q}_i)(\ell_i)$. Then

$$P_M(z) = \sum P_{(S/\mathfrak{q}_i)(\ell_i)}(z).$$

Suppose $\mathcal{Z}(\mathfrak{q}_i)$ is a projective variety of dimension r_i and degree d_i; then its Hilbert polynomial has degree r_i. Only the minimal primes contribute to the leading coefficient of the Hilbert polynomial of M, since the rest have degree less than $r-1$.

Twisting doesn't affect the leading coefficient of the Hilbert polynomial, so that the leading coefficient is

$$\begin{aligned}
\operatorname{lcoeff} P_M(z) &= \operatorname{lcoeff} \sum_{j=1:\mathfrak{q}_j \text{ minimal}}^{q} P_{(S/\mathfrak{q}_i)}(z) \\
&= \sum_j \mu_{\mathfrak{p}_j}(M) \operatorname{lcoeff} P_{(S/\mathfrak{p}_j)}(z) \\
&= \sum_j i(X, H; Z_j) \operatorname{lcoeff} P_{(S/\mathfrak{q}_i)}(z)
\end{aligned}$$

multiply by $(r-1)!$, then

$$\deg(X \cap H) = \sum_j i(X, H; Z_j) \deg(Z_j)$$

as desired. □