12 Hilbert polynomials

12.1 Calibration

Let $X \subset \mathbb{P}^n$ be a (not necessarily irreducible) closed algebraic subset. In this section, we’ll look at a device which measures the way $X$ sits inside $\mathbb{P}^n$.

Throughout this section, let $S = k[X_0, \cdots, X_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$, and let $S(X)$ be the homogeneous coordinate ring of $X$. Then $S(X)$ is a graded module over the graded ring $S$. Define the Hilbert function of $X$ by

$$h_X(\ell) = \dim S(X)_\ell,$$

the dimension of the $\ell^{th}$ graded piece of the coordinate ring of $X$. Recall that this is isomorphic to $S_\ell/I(X)_\ell$.

**Example** $h_{\mathbb{P}^n}(\ell) = \binom{n+\ell}{\ell}$. We saw this before when were examining the Veronese embedding. This means that

$$h_X(\ell) = h_{\mathbb{P}^n}(\ell) - \dim_k I(X)_\ell.$$

**Example** Suppose that $X$ consists of three distinct points in $\mathbb{P}^2$. They either do or don’t live on a line...

- If the points are collinear, then there is a linear relation in $I(X)$, and $h_X(1) = 2 - 1 = 1$.
- If the points are not collinear, then there is not a linear relation in $I(X)$, and $h_X(1) = 2 - 0 = 2$.

Having said that, we have:

**Claim** If $X$ consists of three distinct points in $\mathbb{P}^2$, then $h_X(2) = 3$.

**Proof** For each point $P_i$, let $L_i$ be a linear form which vanishes on $P_i$, but not on the others. Then the product $L_iL_j$ is a quadratic function which vanishes on $P_i$ and $P_j$, but not the other. This gives a surjective map from $S(\mathbb{P}^2)_2$ to the space of functions on $X$, so that $h_X(3) = \binom{2+2}{2} - 3 = 3$. □

In fact, for all $\ell \geq 3$, $h_X(\ell) = 3$, no matter what position the points are. Two things worth pointing out:

- For small $\ell$, $h_X(\ell)$ depends on the arrangement of $X$. 

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• For sufficiently large $\ell$, this difference is erased.

Here is another example. Suppose $X$ is a hypersurface in $\mathbb{P}^n$; say $\mathcal{I}(X) = (F)$, with $F \subset k[X_0, \ldots, X_n]$ of degree $d$. Then $\mathcal{I}(X)_m$ is the polynomials of degree $m$ which are divisible by $F$. This means that “multiplication by $F$” gives an isomorphism between $S_{m-d}$ and $\mathcal{I}(X)_m$, so that
\[ h_X(m) = h_{\mathbb{P}^n}(m) - h_{\mathbb{P}^n}(m-d) \]
Assume $\ell \geq d$; then
\[ \binom{n+\ell}{\ell} - \binom{n+\ell-d}{\ell-d} \]
For instance, if $n = 2$, then
\[ = d\ell - \frac{d(d-3)}{2}. \]
There’s a regularity there which is independent of the curve. The goal of this chapter is to generalize these remarks...

12.2 Algebra

12.2.1 Numerical polynomials

See homework. The point is that a function $h : \mathbb{N} \to \mathbb{Z}$ is called a numerical polynomial if there’s some $P \in \mathbb{Q}[z]$ such that, for $\ell \gg 0$, $h(\ell) = P(\ell)$.

12.2.2 Hilbert polynomials of graded modules

Let $S$ be a graded noetherian ring. A $S$-module $M$ is graded if it comes equipped with a decomposition
\[ M = \oplus M_d \]
such that $S_d M_e \subseteq M_{d+e}$.
If $\ell \in \mathbb{N}$, the twist of $M$ by $\ell$ is the same module with the grading shifted:
\[ M(\ell)_d = M_{d+\ell}. \]
The annihilator of $M$ is
\[ \text{Ann}(M) = \{ x \in S : x \cdot M = 0 \}. \]
If $M$ is graded, this is a homogeneous ideal in $S$. 

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Lemma  Let $M$ be a finitely generated graded module over $S$. There exists a filtration

$$0 \subseteq M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$$

by graded submodules such that for each $i$,

$$M^i / M^{i-1} \cong (S/p_i)(\ell_i)$$

where $p_i \subseteq S$ is a homogeneous prime ideal and $\ell \in \mathbb{Z}$.

Proof  The proof is just like the existence of prime ideal factorizations in noetherian rings.

Given any $M$, let $\mathcal{F}(M)$ be the set of graded submodules of $M$ which admit such a filtration. It’s nonempty, since $(0)$ is in this class. Let $M' \subseteq M$ be a maximal element; it exists, since $M$ is a noetherian module. Consider $M'' = M / M'$. If $M'' = 0$, we’re done.

Otherwise, consider the collection of ideals which are annihilators of homogeneous elements,

$$I = \{ I_m = \text{Ann}(m) : m \in M'' - \{0\} \text{ homogeneous} \}.$$ 

Then each $I_m$ is a proper homogeneous ideal. By noetherianness, we can take a maximal element $I_m$ of $I$.

Claim  $I_m$ is prime.

Suppose $a, b \in S, ab \in I_m, b \not\in I_m$; we’ll show $a \in I_m$. By taking homogeneous components, we may assume $a$ and $b$ are homogeneous. Consider $bm \in M''$. Since $b \not\in I_m, bm \neq 0$. Since $I_m \subseteq bm$, by maximality $I_m = I_{bm}$; but then $abm = 0$, so $a \in I_{bm} = I_m$.

So, $I_m$ is a homogeneous prime ideal, call it $p$. Let $\deg(m) = \ell$. Then the module $N \subseteq M''$ generated by $m$ is isomorphic to $(S/p)(-\ell)$. Lift this:

$$N' \longrightarrow M$$

$$N \cong (S/p)(-\ell) \longrightarrow M''$$

Then $M' \subseteq N', N'/M' \cong (S/p)(-\ell); N'$ has a suitable filtration, which contradicts the maximality of $M'$. Therefore, $M' = M$. □

The prime ideals $\{p_1, \cdots, p_r\}$ which show up should be thought of as the “elementary divisors” of the $S$-module $M$. Among them, we distinguish the minimal ones; these are the minimal primes of $M$.

Lemma  Let $p \subseteq S$ be a homogeneous prime ideal. Then $p|\text{Ann}(M)$ if and only if $p \supseteq p_i$ for one of the minimal primes $p_i$ of $M$. 

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Proof  
\[ p \supseteq \text{Ann}(M) \text{ if and only if } p \text{ annihilates some } M^i / M^{i-1}. \]
But \( \text{Ann}((S/p_i)(\ell)) = p_i \).

Recall that the localization of \( S \) at a prime ideal \( p \) is
\[ S_p = \{ \frac{x}{s} : \deg x = \deg s, s \notin p \}. \]

Lemma  
Let \( p \) be a minimal prime of \( M \), and choose any filtration as above. Then the number of \( i \) for which \( p_i = p \) is the length of \( M_p \) as \( S_p \)-module, where \( S_p \) is the localization.

Proof  
Choose some filtration. For each \( i \) with \( p_i \neq p \), there exists \( a \in p_i \) which is a unit in \( S_p \), so that
\[ M^i_p / M^{i-1}_p \cong S_p \otimes_S (S/p_i) \cong (0). \]
(Use the fact that \( 1 \otimes 1 = (a^{-1} \cdot a) \otimes 1 = a^{-1} \otimes a = 0 \).)
On the other hand, if \( p = p_i \), then
\[ M^i_p / M^{i-1}_p \cong S_p \otimes (S/p) \cong (S/p). \]
Therefore, \( M_p \) is an \( S_p \)-module of the advertised length. \( \square \)

Definition  
The multiplicity of \( M \) at \( p \) is \( \mu_p(M) \), the length of \( M_p \) over \( S_p \).

12.3 Hilbert-Serre theorem

Attached to a graded module is the Hilbert function,
\[ h_M(\ell) = \dim_k M_\ell. \]

Theorem  \([\text{Hilbert-Serre}]\) Let \( S = k[x_0, \cdots, x_n] \) with the standard grading, and let \( M \) be a finitely generated graded \( S \)-module. Then there is a unique polynomial \( P_M(z) \in \mathbb{Q}[z] \) such that, for \( \ell \gg 0 \),
\[ h_M(\ell) = P_M(\ell). \]
Moreover, \( \deg P_M = \dim \mathbb{Z}_{\mathbb{P}^n}(\text{Ann}(M)) \).

Remark  
We assign \( \deg 0 = -1 \), and \( \dim \emptyset = -1 \).
Proof. A homomorphism of graded modules necessarily preserves the grading. Therefore, any short exact sequence of graded modules

\[
0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow 0
\]
yields the equality

\[
h_{M^1}(\ell) + h_{M^2}(\ell) = h_{M^3}(\ell)
\]
for each \(\ell\). After choosing a filtration as above, it suffices to prove the theorem for \((S/p)(\ell)\).

Since \(h_{M}(\ell) = h_{M}(d+\ell)\), it suffices to prove the theorem for \(M = (S/p)\). We distinguish two cases, the first of which is the base case and the second is an inductive step, by induction on \(\text{dim } \mathcal{Z}(\text{Ann}(M))\).

- Suppose \(p = (X_0, \cdots , X_n)\). Then \(h_{M}(\ell) = 0\) for \(\ell > 0\), and thus for \(\ell \gg 0\). Moreover, \(\mathcal{Z}(p) = \emptyset\), so that \(\text{dim } \mathcal{Z}(p) = \text{deg } h_{M} = -1\).

- Otherwise, suppose \(X_i \notin p\). Then – remember that \(\text{deg } x_i = 1\) – we have an exact sequence

\[
0 \longrightarrow M(-1) \overset{x_i}{\longrightarrow} M \longrightarrow M'' = M/x_iM \longrightarrow 0,
\]
and \(h_{M''}(\ell) = h_{M}(\ell) - h_{M}(\ell - 1) = (\Delta h_{M})(\ell - 1)\).

Now, \(\mathcal{Z}(\text{Ann}(M'')) = \mathcal{Z}(p) \cap \mathcal{Z}(x_i)\). The hyperplane section \(\mathcal{Z}(x_i)\) doesn’t contain \(\mathcal{Z}(p)\) (by hypothesis), so that \(\text{dim } \mathcal{Z}(\text{Ann}(M'')) = \text{dim } \mathcal{Z}(p) - 1\). By induction, \(h_{M''}\) is a numerical polynomial, represented by a polynomial \(P_{M''}\) of degree \(\text{dim } \mathcal{Z}(\text{Ann}(M''))\). Since \((\Delta h_{M})\) is a numerical polynomial of degree \(\text{dim } \mathcal{Z}(\text{Ann}(M)) - 1\), \(h_{M}\) is a numerical polynomial of degree \(\text{dim } \mathcal{Z}(\text{Ann}(M))\).

The only missing ingredient is the lemma promised in the homework:

Lemma. Let \(f : \mathbb{Z} \rightarrow \mathbb{Z}\) be any function. Suppose that there exists a numerical polynomial \(q\) such that \(\Delta(f)(n) = q(n)\) for all \(n \gg 0\). Then there exists a numerical polynomial \(p\) such that \(f(n) = p(n)\) for all \(n \gg 0\).

Proof. By the homework, there exists a polynomial \(Q(z) = \sum_{j=0}^{r} c_j \binom{z}{j}\) such that, for \(n \gg 0\), \(Q(n) = q(n)\). Let

\[
P(z) = \sum_{j=0}^{r} c_j \binom{z}{j+1}.
\]

Then for \(n \gg 0\), \(\Delta(P) = \Delta(f) = Q(n) = q(n)\), so that \(\Delta(P - f)(n) = 0\) for \(n \gg 0\). Therefore, \((P - f)(n) = a\) for some \(a \in \mathbb{Z}\) and all \(n \gg 0\), so that \(f(n) = P(n) - a\) is a numerical polynomial.
12.4 Geometry of the Hilbert polynomial

**Definition** Suppose $Y \subset \mathbb{P}^n$. The Hilbert polynomial $P_Y$ of $Y$ is the Hilbert polynomial of its homogeneous coordinate ring $S(Y)$. Note that $\deg P_Y = \dim Y$. The degree of $Y$ is $(\dim Y)!$ times the leading coefficient of $P_Y$.

**Remark** Since $P_Y$ is a numerical polynomial, it has a binomial representation

$$P_Y(z) = \sum_{j=0}^{r} c_j \binom{z}{j}.$$  \hspace{1cm} (1)

with $c_j \in \mathbb{Z}$; then $\deg Y = c_r$.

**Lemma** If $Y \subset \mathbb{P}^n$ is nonempty, then $\deg(Y) \in \mathbb{N}$.

**Proof** Since $Y \neq \emptyset$, $P_Y$ is a nonzero polynomial of degree $r = \dim Y$. Then $\deg Y = c_r \in \mathbb{Z}$, as above. It’s positive since there are functions on $Y$ of arbitrarily large degree, so that $h_Y(\ell) > 0$ infinitely often.

**Lemma** Suppose $Y \subset \mathbb{P}^n$, $Y = Y_1 \cup Y_2$, $\dim Y_1 = \dim Y_2 = r$, $\dim(Y_1 \cap Y_2) < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$.

**Proof** Let $I_j = \mathcal{I}(Y_j)$, and let $I = I_1 \cap I_2 = \mathcal{I}(Y)$. Note that $\mathcal{I}(Y_1 \cap Y_2) = \sqrt{(I_1 + I_2)}$. There is an exact sequence

$$0 \longrightarrow S/I \longrightarrow S/I_1 \oplus S/I_2 \longrightarrow S/(I_1 + I_2) \longrightarrow 0$$

Since $\dim(Y_1 \cap Y_2) < r$, $\deg S/(I_1 + I_2) < r$, and the leading coefficient of $S/I$ is the sum of the leading coefficients of those of $S/I_1$ and $S/I_2$.

**Lemma** $\deg \mathbb{P}^n = 1$

**Proof** Use the earlier calculation of $P_{\mathbb{P}^n}(z)$.

**Lemma** If $H \subset \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree $d$, then $\deg H = d$.
Proof  Suppose $I(H) = (F)$, $F$ homogeneous of degree $d$. There’s a diagram of graded modules

$$
0 \longrightarrow S(-d) \overset{F}{\longrightarrow} S \longrightarrow S(H) = S/(F) \longrightarrow 0
$$

Then

$$
h_H(\ell) = h_{P^n}(\ell) - h_{P^n}(\ell - d)
$$

$$
h_H(z) = h_{P^n}(z) - h_{P^n}(z - d)
$$

$$
= \left( \frac{z + n}{n} \right) - \left( \frac{z - d + n}{n} \right)
$$

$$
= \frac{d}{(n-1)!} z^{n-1} + \cdots
$$

and the degree of $H$ is $d$. 

Remark  The degree depends not just on the variety, but on the way it sits in the ambient projective space. Do an example of the $d$-uple embedding?

12.5 Intersection theory

Let $X \subset \mathbb{P}^n$ be a projective variety, not necessarily irreducible, of pure dimension $r$. Let $H \subset \mathbb{P}^n = Z(F)$ be some hypersurface – choose $F$ reduced. By the principal ideal theorem,

$$
X \cap H = Z_1 \cup \cdots \cup Z_m,
$$

where $\dim Z_j = r - 1$. Each $Z_j$ corresponds to a homogeneous prime ideal $p_j = I_X(Z_j)$.

Let $S = k[X_0, \ldots, X_n]$. Then we have homogeneous coordinate rings

$$
S(X) = S/I(X)
$$

$$
S(H) = S/I(H) = S/(F)
$$

$$
S(X \cap H) = S/\sqrt{(I(X) + I(H))}
$$

Note that, as $S$-module, $\text{Ann}(S(X \cap H)) = I(X) + I(H)$, and the minimal primes of this module are simply the $p_1, \cdots, p_m$.

Definition  The intersection multiplicity of $X$ and $H$ along $Z_j$ is

$$
i(X, H; Z_j) = \mu_{p_j}(S/(I(X) + I(H))).
$$
Example  Consider the elliptic curve \( E = \mathbb{P}(Y^2Z - X^3 + XZ^2) \subset \mathbb{P}^2 \), and assume we’re not in characteristic three. Let \( H \) be the hyperplane \( H = Z(Y) \). Then we compute intersection multiplicities by looking at the module

\[
M = \frac{k[X,Y,Z]}{(Y^2Z - X^3 + XZ^2, Y)}
\]

\[
= \frac{k[X,Y,Z]}{(X^3 - XZ^2, Y)}
\]

\[
\cong \frac{k[X,Z]}{(X^3 - XZ^2)}
\]

\[
\cong \frac{k[X,Z]}{X(X - Z)(X + Z)}
\]

Note that as a \( k \)-module, this has dimension three. There are three maximal ideals of \( k[X, Z] \) which contain the ideal \( (X^3 - XZ^2) \), namely, \( (X) \), \( (X - Z) \), \( (X + Z) \). These are the three (minimal) prime ideals associated with this module. Let, say, \( p = (X - Z) \). If we localize \( k[X,Y] \) at \( p \), then we invert (in particular) \( X \) and \( X + Z \), so that

\[
M_p \cong k[X,Z]/(X - Z)
\]

has length one as \( M_p \)-module. The same is true for the other ideals.

Example  Same elliptic curve, but now consider the hyperplane \( W = Z(X - Z) \). Then the module in question is

\[
N = \frac{k[X,Y,Z]}{(Y^2Z - X^3 + XZ^2, X - Z)}
\]

\[
= \frac{k[X,Y]}{(Y^2X - X^3 + X^3)}
\]

\[
= \frac{k[X,Y]}{(Y^2X, X^3)}
\]

The minimal primes of this module are \( (Y) \) and \( (X) \), corresponding to the (projective) points \( [1, 0, 1] \) and \( [0, 1, 0] \) (the “point at infinity”), respectively. The multiplicities at these points are two and one, respectively.

Theorem  With all notation as above (especially, \( X \not\subset H \)),

\[
\sum_{j=1}^{g} i(X,H; Z_j) \cdot \deg(Z_j) = (\deg X)(\deg H).
\]
Proof Suppose $\deg H = d$, $H = \mathcal{Z}(F)$, $F$ reduced. Let $M = S/(\mathcal{I}(X) + \mathcal{I}(H))$. As before, we have an exact sequence

$$0 \longrightarrow S(X)(-d) \xrightarrow{F} S(X) \xrightarrow{} M \xrightarrow{} 0$$

so that

$$P_M(z) = P_X(z) - P_X(z - d).$$

Consider a (maximal) filtration $M^0 \subset M^1 \subset \cdots \subset M^q = M$, with quotients $M^i/M^{i-1} \cong (S/q_i)(\ell_i)$. Then

$$P_M(z) = \sum P_{(S/q_i)(\ell_i)}(z).$$

Suppose $\mathcal{Z}(q_i)$ is a projective variety of dimension $r_i$ and degree $d_i$; then its Hilbert polynomial has degree $r_i$. Only the minimal primes contribute to the leading coefficient of the Hilbert polynomial of $M$, since the rest have degree less than $r - 1$.

Twisting doesn’t affect the leading coefficient of the Hilbert polynomial, so that the leading coefficient is

$$\text{lcoeff } P_M(z) = \text{lcoeff } \sum_{j=1: q_j \text{ minimal}}^q P_{(S/q_i)}(z)$$

$$= \sum_j \mu_{q_j}(M) \text{lcoeff } P_{(S/p_j)}(z)$$

$$= \sum_j i(X, H; Z_j) \text{lcoeff } P_{(S/q_i)}(z)$$

multiply by $(r - 1)!$, then

$$\deg(X \cap H) = \sum_j i(X, H; Z_j) \deg(Z_j)$$

as desired. \qed