Homework 9
Due: Friday, November 13

1. Let $\Delta(z) = \sum_m \tau(m) q^m$. Use the theory from class to (quickly) show that if $\gcd(m,n) = 1$, then $\tau(mn) = \tau(m)\tau(n)$.

2. Recall that $\mathcal{S}_{12} = \mathcal{S}_{12}(\Gamma(1))$, the space of cusp forms of weight twenty four, has dimension two. Compute the eigenvalues of the action of T_2 on $\mathcal{S}_{12}$, as follows.

   (a) Write down a basis $\{f_1, f_2\}$ for $\mathcal{S}_{12}$.

   (b) Find the first few Fourier coefficients of each f_i:
$$f_i(z) = \sum_{0 \le j \le 3} c_i(j) q^j + \mathcal{O}(q^4).$$

3. (2) *continued.*

   (a) Compute the first few Fourier coefficients of $T_2 f_1$ and $T_2 f_2$.

   (b) Find a matrix representing the effect of T_2 on $\mathcal{S}_{12}$. What are its eigenvalues?

4. The Gamma function is defined by
$$\Gamma(s) = \int_0^\infty \exp(-t) t^{s-1}\, dt. \tag{1}$$
It turns out that the right-hand side is holomorphic on $\mathrm{Re}(s) > 0$; assume this.

   (a) Show that, if $\mathrm{Re}(s) > 0$, then $\Gamma(s+1) = s\Gamma(s)$. (HINT: *Integrate by parts.*)

   (b) Show that $\Gamma(n+1) = n!$.

5. (4) *continued.* Consider the functions
$$\begin{aligned}
F_1(s) &= \frac{1}{s}\Gamma(s+1) \\
F_2(s) &= \frac{1}{s}\frac{1}{s+1}\Gamma(s+2) \\
F_3(s) &= \frac{1}{s}\frac{1}{s+1}\frac{1}{s+2}\Gamma(s+3) \\
&\vdots \\
F_N(s) &= \frac{1}{s\cdot(s+1)\cdot(s+2)\cdot\dots\cdot(s+N-1)}\Gamma(s+N) \\
&\vdots
\end{aligned}$$

   (a) Show that $F_N(s)$ is meromorphic on $\mathrm{Re}(s) > -N$.

   (b) Show that if $N > M$, then F_N and F_M agree on $\mathrm{Re}(s) > -M$.

   (c) Explain why $\Gamma(s)$ admits a meromorphic continuation to $\mathbb{C}$; it has simple poles at negative integers, and is holomorphic elsewhere.