Homework 8

Due: Friday, November 6

1. Suppose $\alpha : \mathbb{N} \to \mathbb{C}$ is (completely) multiplicative;

$$\alpha(mn) = \alpha(m)\alpha(n).$$

Suppose it grows sufficiently slowly that

$$L(\alpha, s) := \sum_{n\geq 1} \alpha(n) n^{-s}$$

converges on some half-plane of $\mathbb{C}$. Show that there is a factorization

$$L(\alpha, s) = \prod_p \frac{1}{1 - \alpha(p)p^{-s}}.$$

2. Now suppose that α is weakly, but not completely, multiplicative. Set

$$L_p(\alpha, s) = \sum_{m\geq 0} \alpha(p^m) p^{-ms}.$$

   (a) Show that there is an equality

   $$\sum_{n\geq 1} \alpha(p)\alpha(p^n)p^{-ns} = \alpha(p)(L_p(\alpha, s) - 1).$$

   (b) Similarly, express

   $$\sum_{n\geq 1} \alpha(p^{n+1})p^{-ns} \text{ and } \sum_{n\geq 1} p^{11}\alpha(p^{n-1})p^{-ns}$$

   in terms of $L_p(\alpha, s)$.

   (c) Suppose that, for each $n \geq 1$,

   $$\alpha(p)\alpha(p^n) = \alpha(p^{n+1}) + p^{11}\alpha(p^{n-1}).$$

   Show that

   $$L_p(\alpha, s) = \frac{1}{1 - \alpha(p)p^{-s} + p^{11-2s}}.$$

3. Write $1 - \alpha(p)T + p^{11}T^2 = (1 - \gamma T)(1 - \gamma' T)$ for complex numbers γ, γ', and suppose that $\alpha(p)$ is real. Show that the following are equivalent:

   (a) $|\alpha(p)| \leq 2p^{11/2}$;

   (b) $|\gamma| = |\gamma'| = p^{11/2}$;

   (c) $\gamma' = \overline{\gamma}$.

4. One (ad hoc) way of defining Hecke operators is as follows. Suppose $f \in \mathcal{M}_k = \mathcal{M}_k(\Gamma(1))$, and $m \in \mathbb{N}$. If f has Fourier expansion

$$f(z) = \sum a_n q^n$$

then

$$T(m)f = m^{2k-1} \sum_{ad=m, a,d>0} \frac{1}{d^{2k}} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

(a) Show that

$$\begin{aligned} T(m)f &= \sum_{d|m, d>0} (m/d)^{2k-1} \sum_{n \geq 0, d|n} a_n q^{mn/d^2} \\ &= \sum_{n \geq 0} \left(\sum_{r|(m,n), r>0} r^{2k-1} a_{mn/r^2} \right) q^n. \end{aligned}$$

(b) Conclude that for all m and n, $T(m)T(n) = T(n)T(m)$.

5. Use the previous problem to show that, for each m,

$$T(m)\Delta = \tau(m)\Delta.$$