Homework 7

Due: Friday, October 23

The first three problems sketch another derivation of the series expansion for

$$\begin{aligned} g(z) &= \pi\cot(\pi z) \\ &= \pi i \frac{\exp(2\pi i z)+1}{\exp(2\pi i z)-1}. \end{aligned}$$

Let

$$\begin{aligned} f(z) &= \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{z-n} \\ &= \frac{1}{z} + \sum_{1\le n\le N}\left(\frac{1}{z-n}+\frac{1}{z+n}\right) \\ &= \frac{1}{z} + \sum_{n\ge 1} \frac{2z}{z^2-n^2}. \end{aligned}$$

The sum is absolutely convergent on compact sets, and $f(z)$ is defined away from $\mathbb{Z}$.

1. Prove the following facts about $f(z)$.
   (a) If $z \notin \mathbb{Z}$, then $f(z) = f(z+1)$.
   (b) $f(z) = \frac{1}{z} + f_0(z)$, where $f_0(z)$ is analytic near 0.
   (c) $f(z)$ has simple poles at the integers, and no other singularities.
2. Prove the following facts about $g(z)$.
   (a) If $z \notin \mathbb{Z}$, then $g(z) = g(z+1)$.
   (b) $g(z) = \frac{1}{z} + g_0(z)$, where $g_0(z)$ is analytic near 0.
   (c) $g(z)$ has simple poles at the integers, and no other singularities.
3. Consider the difference function
   $$h(z) = g(z) - f(z).$$
   (a) Show that $h(z)$ is entire, i.e., holomorphic on $\mathbb{C}$.
   (b) One can show that $f(z)$ is bounded on
   $$S = \{z \in \mathbb{C} : |\mathrm{Re}(z)| \le \frac{1}{2} \text{ and } |\mathrm{Im}(z)| > 1\}.$$
   Explain how to conclude that $h(z)$ is constant.

4. Show that, for $k \geq 2$,

$$\sum_{m \in \mathbb{Z}} \frac{1}{(m+z)^k} = \frac{1}{(k-1)!}(-2i\pi)^k \sum_{n \geq 1} n^{k-1} q^n \tag{1}$$

where $q(z) = \exp(2\pi i z)$. (HINT: *Start with two different expansions for* $\pi \cot(\pi z)$, *and take derivatives.*)

5. Use (1) to show that, for $k \geq 2$,

$$G_k(z) = 2\zeta(2k) + 2\frac{(2i\pi)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

where $q = \exp(2\pi i z)$ and, for a natural number N and a nonnegative integer r,

$$\sigma_r(N) = \sum_{d|N} d^r.$$

6. For $k \geq 2$, let

$$\begin{aligned} E_k(z) &= \frac{G_k(z)}{2\zeta(2k)} && (2) \\ &= 1 + \gamma_k \sum_{n \geq 1} \sigma_{2k-1}(n) q^n && (3) \end{aligned}$$

where

$$\gamma_k = (-1)^k \frac{4k}{B_k}. \tag{4}$$

(a) Compute γ_k for $2 \leq k \leq 8$ (use any method or tool you like).

(b) Show that there are identities of modular forms

$$\begin{aligned} E_2^2 &= E_4 \\ E_2 E_3 &= E_5 \end{aligned}$$

(HINT: $\dim \mathcal{M}_4 = \dim \mathcal{M}_5 = 1$.)

7. Show that, for each n,

$$\begin{aligned} \sigma_7(n) &= \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) \\ 11\sigma_9(n) &= 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(n)\sigma_5(n-m). \end{aligned}$$