Homework 6

Due: Friday, October 16

The first half is easy; the second half will only make sense after class on Monday.

1. For a positive integer N, we defined

$$\begin{aligned} Y_N = \{(E, P, Q) : & \ E/\mathbb{C} \text{ an elliptic curve} \\ & \{P, Q\} \text{ a basis for } E[N] \\ & e_N(P, Q) = \exp(2\pi i/N)\} \end{aligned}$$

   and a map

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & Y_N \\ \tau & \longmapsto & (E_\tau, \frac{\tau}{N}, \frac{1}{N}) \end{array}$$

   which yielded a bijection $Y(\Gamma(N)) \xrightarrow{\sim} Y_N$.

   Find a map ϕ which makes the following diagram commute; prove your answer is correct.

$$\begin{array}{ccc} \Gamma(mn)\backslash\mathbb{H} & \longrightarrow & Y_{mn} \\ \downarrow & & \vdots \phi \\ \Gamma(n)\backslash\mathbb{H} & \longrightarrow & Y_n \end{array}$$

2. Recall that we have defined:

$$G_k(\tau) = f_{2k}(\Lambda_\tau)$$

   where

$$\begin{aligned} f_k(\Lambda) = \sum \lambda \in \Lambda \smallsetminus \{0\} \frac{1}{\lambda^k}; g_2(\tau) & = 60G_2(\tau) \\ g_3(\tau) = 140G_3(\tau) & \end{aligned}$$

   Quickly explain why:

   (a) $f_k(\tau)$ is identically zero if k is odd;

   (b) If $i\Lambda_\tau = \Lambda_\tau$, then $g_3(\tau) = 0$.

   (c) If $\rho\Lambda_\tau = \Lambda_\rho$, then $g_2(\tau) = 0$. (Here, $\rho = \exp(2\pi i/3)$.)

3. Let f be a meromorphic modular form of weight $2k$ for $\Gamma(1)$. Show that

$$\operatorname{ord}_{i\infty}(f) + \frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \sum_Q \operatorname{ord}_Q(f) = \frac{k}{6},$$

where the sum is over all Q in our (open) fundamental domain. (HINT: *Just specialize our general result on zeros of modular forms.*)

4. (a) G_2 has a simple zero at $z = \rho$, and no others.
   (b) G_3 has a simple zero at $z = i$, and no others.
   (c) Δ has a simple zero at ∞.

5. Let $\mathcal{M}_k = \mathcal{M}_k(\Gamma(1))$ be the space of holomorphic modular forms of weight $2k$, and let $\mathcal{S}_k = \mathcal{S}_k(\Gamma(1)) \subseteq \mathcal{M}_k$ be the space of cuspidal holomorphic modular forms; by definition, $\mathcal{S}_k$ is the kernel of the map

$$\begin{aligned} \mathcal{M}_k &\longrightarrow \mathbb{C} \\ f &\longmapsto f(i\infty). \end{aligned}$$

   (a) For $1 \le k \le 5$, show that G_k is a basis for $\mathcal{M}_k$, and that $\mathcal{S}_k = 0$.
   (b) For any k, show that multiplication by Δ gives an isomorphism

$$\begin{aligned} \mathcal{M}_k &\longrightarrow \mathcal{S}_{k+6} \\ f &\longmapsto \Delta f \end{aligned}$$

As a consequence of study of modular forms we will find that, for a positive integer k,

$$\#\{a, b \in \mathbb{Z}_{\ge 0} : 2a + 3b = k\} = \begin{cases} \lfloor k/6 \rfloor & k \equiv 1 \bmod 6 \\ \lfloor k/6 \rfloor + 1 & k \not\equiv 1 \bmod 6 \end{cases}.$$

Feel free to prove this directly, if you're so inclined.