
Homework 5

Due: Friday, October 9

The first two questions are very easy, but useful.

1. Prove the following two trivial, but useful, observations:
   (a) Show there are no nonconstant modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of odd weight.
   (b) If $\Gamma \subset \Gamma(1)$, the width of the cusp at ∞ is the smallest n such that $T^n \in \Gamma$.
   If α is any other cusp, its *width* is the smallest n such that $T^n \in \delta^{-1}\Gamma\delta$, where $\delta \in \Gamma(1)$ satisfies $\delta\alpha = \infty$.
   Suppose Γ is normal in $\Gamma(1)$. Show that all cusps have the same width.

2. Fix an integer $N \geq 2$, and consider $\Gamma(N)$; let $\overline{\Gamma}(N)$ be its image mod $\pm$ id. Note that $\Gamma(N)$ is normal in $\Gamma(1)$.
   (a) Show that $\Gamma(N)$ has no elliptic points in $\mathbb{H}$. (HINT: *For* $\overline{\Gamma}(1) = \mathrm{PSL}_2(\mathbb{Z})$*, any elliptic point has stabilizer conjugate to either* $\langle S \rangle$ *or* $\langle ST \rangle$*.*)
   (b) Compute the width of the cusp at ∞.

3. *Continuation of previous problem.*
   (a) Suppose a group G acts transitively on a set X. Let $H \subseteq G$ be a normal subgroup of finite index. Let $x_0 \in X$ be any point. Show that the number of orbits of H, acting on X, is
   $$\frac{[G : H]}{[\mathrm{Stab}_G(x_0) : \mathrm{Stab}_H(x_0)]}.$$
   (b) Compute the number of cusps for $\Gamma(N)$.
   (c) Give a formula for the genus of the modular curve $X(N)$. (HINT: *Use the theorem from class: Let* $d_N = [\overline{\Gamma}(1) : \overline{\Gamma}]$*. Let* ν_∞ *be the number of inequivalent cusps; let* ν_2 *be the number of inequivalent elliptic points of order* 2*; let* ν_3 *be the number of inequivalent elliptic points of order* 3*. Then the genus of* $X(\Gamma)$ *is*
   $$1 + \frac{d}{12} - \frac{1}{4}\nu_2 - \frac{1}{3}\nu_3 - \frac{1}{2}\nu_\infty.$$
   .)

4. Let $\Gamma \subset \Gamma(1)$ have finite index. Let $\omega = f(\tau)d\tau$ be a meromorphic differential on $\mathbb{H}$.
   (a) Suppose $\gamma \in \Gamma \subset \Gamma(1)$. Calculate $\gamma^*\omega$.
   (b) Show that ω is invariant under Γ if and only if f is a meromorphic modular form of weight 2.

5. A k-fold differential on a Riemann surface is given locally by an expression of the form

$$\omega = f(z) \cdot (dz)^k;$$

it transforms as

$$g^*\omega = g^* f (d\, g(z))^k$$
$$f(g(z))g'(z)^k(dz)^k.$$

Verify that meromorphic modular forms of weight $2k$ for Γ correspond to meromorphic k-fold differentials on $\Gamma\backslash\mathbb{H}$.

In fact, they are meromorphic at the cusps, and actually correspond to forms on $\Gamma\backslash\mathbb{H}^*$.