Homework 4
Due: Friday, October 2

1. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\Gamma' = \langle S, T \rangle$. The goal of this problem is to give an alternate proof that, setting $\Gamma' = \langle S, T \rangle$, we have $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$.

   Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

   For a matrix A, let A_{ij} denote the entry in row i, column j.

   (a) Suppose $c = 0$. Show that $\gamma \in \Gamma'$.
   Henceforth, assume $c \neq 0$.

   (b) Show that we may assume $|\gamma_{11}| \geq |\gamma_{12}|$.

   (c) Show there exists some q such that

   $$\left|(T^{-q}\gamma)_{11}\right| \leq \left|(T^{-q}\gamma)_{21}\right|.$$

   (HINT: *Write $a = cq + r$, $0 \leq r < |c|$.*)

   (d) *Iterate.*

2. Let X and Y be compact Riemann surfaces, and let $\phi : X \to Y$ be a nonconstant holomorphic map. If $P \in X$, the *multiplicity* of ϕ at P may be defined as follows: Let w be a local coordinate on X centered at P (so that $w(P) = 0$), and let z be a local coordinate on Y centered at Q. Locally, ϕ is given by a function $z = f(w)$; then $\mu_P(\phi) = \mathrm{ord}_{w=0} f$.

   Consider the local meromorphic differential dz on Y. Calculate

   $$\mathrm{ord}_P f^* dz;$$

   your answer should be in terms of $\mu_P(\phi)$.

   (HINT: *By HW3, you can choose local coordinates so that ϕ is given by*

   $$z = w^{\mu_P(\phi)}.$$

)

3. If $\phi : X \to Y$ is a nonconstant holomorphic map of compact Riemann surfaces, there is an induced map

   $$\mathrm{Div}(Y) \xrightarrow{\phi^*} \mathrm{Div}(X)$$

   where

   $$\phi^* Q = \sum_{P \in \phi^{-1}(Q)} \mu_P(\phi) P$$

and we extend by linearity:

$$\phi^*(\sum_Q n_Q Q) = \sum_Q n_Q \phi^* Q.$$

It turns out that, if $\deg \phi = d$, then for each $Q \in Y$ we have

$$\deg(\phi^* Q) = d.$$

(a) Now suppose that ω is a nonconstant meromorphic differential on Y, with no zero or pole in the branch locus of ϕ. Show that

$$\deg(\phi^*\omega) = d \deg(\omega) + \sum_{P \in X} (\mu_P(\phi) - 1).$$

(b) Deduce the Riemann-Hurwitz formula. (HINT: *Remember that if Z is a compact Riemann surface and if ω_Z is a nonconstant meromorphic differential, then* $\deg(\omega_Z) = 2g_Z - 2$.)

4. Recall that

$$\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$$
$$\Gamma(N) = \ker\left(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N)\right).$$

Give an explicit formula for the index

$$[\Gamma(1) : \Gamma(N)].$$

If you like, you may proceed as follows.

(a) Explain why it suffices to calculate $\#\mathrm{GL}_2(\mathbb{Z}/N)$.

(b) Explain why it suffices to calculate, for a prime ℓ, $\#\mathrm{GL}_2(\mathbb{Z}/\ell^r)$.

(c) Explain how to quickly calculate $\#gl_2(\mathbb{Z}/\ell^r)$ in terms of $\#\mathrm{GL}_2(\mathbb{Z}/\ell)$

(d) Calculate $\#gl_2(\mathbb{Z}/\ell)$.

(e) Conclude with a formula for $[\Gamma(1) : \Gamma(N)]$.