Homework 3

Due: Friday, September 25

1. Let $f(z)$ be meromorphic on an open subset $U \subset \mathbb{C}$. Show that, for each $z_0 \in U$,

$$\operatorname{res}_{z_0} \frac{f'(z)}{f(z)} = \operatorname{ord}_{z_0} f(z).$$

2. Suppose f is holomorphic in some neighborhood of z_0, and that $\operatorname{ord}_{z_0} f(z) = k \geq 1$.

   (a) Show that there exists a function $g(z)$, holomorphic and nonvanishing on some open neighborhood of z_0, such that on this neighborhood we have

   $$f(z) = (z\, g(z))^k.$$

   You may use the following result:

   Theorem Suppose $\Omega \subset \mathbb{C}$ is open, connected and simply connected and $0 \notin \Omega$. Then there is a branch of the logarithm on Ω, i.e., a holomorphic function $L(z)$ on Ω such that $\exp(L(z)) = z$.

   (b) Show that, for any sufficiently small neighborhood of z_0, $zg(z)$ is one-to-one onto its image.

   (c) Show that, for any sufficiently small punctured neighborhood of z_0, $f(z)$ is k-to-one onto its image.

3. Let z be the usual coordinate on (the finite part of) $\hat{\mathbb{C}}$, and consider the meromorphic differential $\omega = dz$. Show that its divisor is

$$\operatorname{div}(\omega) = -2(\infty).$$

(HINT: *A coordinate on* $\hat{\mathbb{C}} - \{0\}$ *is* w, *where* $w = \frac{1}{z}$ *on* $\hat{\mathbb{C}} - \{0, \infty\}$. *Express* ω *in terms of* w.)

4. (a) As in HW1, problem 2.5, (re)calculate the stabilizer

   $$K := \operatorname{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i).$$

   Verify that K is compact, and that every finite subgroup of K is cyclic.

   (b) Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup, and let $z \in \mathbb{H}$ be a point. Show that

   $$\operatorname{Stab}_\Gamma(z)$$

   is a finite, cyclic group.

   (HINT: *There exists some* $\alpha \in \mathrm{SL}_2(\mathbb{R})$ *such that* $\alpha(z) = i$ *(why?). Now* $\operatorname{Stab}_\Gamma(z) \cong K \cap (\alpha^{-1}\Gamma\alpha)$.)