Homework 2

Due: Friday, September 11

1. Consider the unit sphere $X = \{(a,b,c) : a^2+b^2+c^2 = 1\} \subset \mathbb{R}^3$. Let $N = (0,0,1)$, $S = (0,0,-1)$, $U_N = X - \{N\}$, $U_S = X - \{S\}$. Consider the following three charts on X:

$$U_N \xrightarrow{\phi_N} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 + ib_0}{1 - c_0}$$

$$U_S \xrightarrow{\phi_S} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 + ib_0}{1 + c_0}$$

$$U_S \xrightarrow{\psi_S} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 - ib_0}{1 + c_0}$$

   (a) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = (\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}).$$

   Calculate $\phi_S^{-1}(z)$ and $\psi_S^{-1}(z)$.

   (b) Among the three charts $\{(U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S)\}$, one pair is compatible and the other two are not. Which is which? Why? (HINT: *Remember that a function f is holomorphic if and only if $\partial_{\bar{z}} f = 0$; colloquially, a function is holomorphic if it doesn't involve any $\bar{z}$'s.*)

2. Fix $n \geq 1$. (Complex) projective n-space is defined as

$$\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \smallsetminus \{0\}) / \sim,$$

   where

$$(a_0, \cdots, a_n) \sim (\lambda a_0, \lambda a_1, \cdots, \lambda a_n)$$

   for each $\lambda \in \mathbb{C}^\times$. The equivalence class of $(a_0, \cdots, a_n)$ in $\mathbb{P}^n$ will be denoted $[a_0, \cdots, a_n]$.

   Remark: We could also define it as the quotient of $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by the action of the multiplicative group $\mathbb{G}_m$; this already shows that $\mathbb{P}^n$ is irreducible and Hausdorff.

   For $0 \leq i \leq n$, let

$$H_i = \{[a_0, \cdots, a_n] : a_i = 0\}$$
$$U_i = \mathbb{P}^n \smallsetminus H_i.$$

and define a chart

$$U_i \xrightarrow{\phi_i} \mathbb{C}^n$$

$$[a_0, \cdots, a_n] \longmapsto \left(\frac{a_0}{a_i}, \cdots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \cdots, \frac{a_n}{a_i} \right).$$

(a) Prove that ϕ_i is well-defined, i.e., that it is independent of the choice of representative for $[a_0, \cdots, a_n]$, and that ϕ_i is an inclusion.

(b) Prove that $\{\phi_i\}_{0 \leq i \leq n}$ is a compatible family of analytic charts on $\mathbb{P}^n$, and thus gives $\mathbb{P}^n$ the structure of a complex manifold.

3. Consider the map

$$\mathbb{P}^1 \xrightarrow{\alpha} \mathbb{R}^3$$

$$[z_0, z_1] \longmapsto \left(\frac{2\operatorname{Re}(z_1 \overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{2\operatorname{Im}(z_1 \overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{|z_1|^2 - |z_0|^2}{|z_1|^2 + |z_0|^2} \right. . \tag{1}$$

(a) Show that this really is a function on $\mathbb{P}^1$, i.e., if $\lambda \in \mathbb{C}^\times$, then $\alpha([\lambda z_0, \lambda z_1]) = \alpha([z_0, z_1])$.

(b) Show that the image of α is the unit sphere $a^2 + b^2 + c^2 = 1$. (In fact, α is a homeomorphism.) (HINT: *Remember that for any* $w \in \mathbb{C}$, $\operatorname{Re}(w) = \frac{w + \overline{w}}{2}$, $\operatorname{Im}(w) = \frac{w - \overline{w}}{2i}$, *and* $|w|^2 = w\overline{w}$.)

4. Endow S^2 with the complex structure of given in Problem (1), and $\mathbb{P}^1$ with the complex structure from Problem (2). Show that the map α in (1) is holomorphic.

5. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$.

(a) Show that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ stabilizes every element of $\mathbb{H}$. *Therefore, the action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$ *factors through the quotient* $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \pm \mathrm{id}$.

(b) Compute the stabilizer $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(i)$.

(c) Find some $P \in \mathbb{H}$ such that $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(P) = \pm \mathrm{id}$.