1 Review: Functions of a single variable

1.1 Analytic functions

Suppose $z_0 \in \mathbb{C}$, U some open neighborhood of z_0, f defined on U. Then f is called analytic, or differentiable, or holomorphic at z_0 if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Cauchy-Riemann equations Think of f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$; $f(z) = u(x,y) + iv(x,y)$. Then $f'(z_0)$ exists if and only if (u and v have continuous first derivatives and)

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned}$$

the Cauchy-Riemann equations

Formal/algebraic version On $\mathbb{C} \cong \mathbb{R}^2$, we have the real coordinates x and y; then $z = x + iy$. The conjugate is $\overline{z} = x - iy$. So, we can also use z and $\overline{z}$ as coordinates on $\mathbb{C}$; then

$$\begin{aligned} x &= \frac{z + \overline{z}}{2} \\ y &= \frac{z - \overline{z}}{2i} \end{aligned}$$

Think of f a function of the *real* variables $f(x,y)$ (abuse of notation). Formally, we have

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \\ &= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{1}{2i} \end{aligned}$$

and define the operator

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} \\ &= \frac{\partial f}{\partial x}\frac{1}{2} + \frac{\partial f}{\partial y}\frac{-1}{2i} \\ \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\end{aligned}$$

Then the Cauchy-Riemann equations are equivalent to:

$$\partial_{\overline{z}} f = 0.$$

Example $\overline{z}$ is not holomorphic.

Theorem Suppose f is analytic everywhere inside and on a simple closed positive contour C. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z_0}dw.$$

Here is another, equivalent way of phrasing this:

Variant Suppose f is analytic on an open set containing $\overline{N_r(z_0)}$. Then for each $z \in N_r(z_0)$,

$$f(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r} \frac{f(w)}{w - z}dw.$$

This has a number of important corollaries.

Corollary Same hypotheses; then f has derivatives of all orders at z_0, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w - z_0)^{n+1}}dw.$$

Lemma Suppose f analytic inside and on a circle C_R centered at z_0 of radius R. Let $M_R = \max_{z\in C_R}|f(z)|$. Then

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M_R}{R^n}.$$

This follows immediately from the representation theorem;

$$\begin{aligned}
\left|f^{(n)}(z)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw\right| \\
&\leq \frac{n!}{2\pi}\int_C \left|\frac{f(w)}{(w-z)^{n+1}}\right||dw| \\
&\leq \frac{n!}{2\pi}\int_C \frac{M_R}{R^{n+1}}|dw| \\
&= \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}2\pi R \\
&= \frac{n!M_R}{R^n}.
\end{aligned}$$

□

Theorem [Liouville] If f is entire and bounded, then f is constant.

Proof Suppose that $f(z) \leq M$ for all z. Then for each z and each $R > 0$, we have

$$|f'(z)| \leq \frac{M}{R}.$$

Therefore, $f'(z) = 0$ for each z, and f is constant. □

Get a series representation:

Theorem Supppose that f is analytic throughout a disk $|z - z_0| < R$. Then $f(z)$ has the power series representation

$$\begin{aligned}
f(z) &= \sum_{n\geq 0} a_n(z-z_0)^n \\
a_n &= \frac{f^{(n)}(z_0)}{n!}
\end{aligned}$$

for $|z - z_0| < R$.

The proof uses the Cauchy representation for the derivatives of a function.

1.2 Orders, residues

There are series developments even for functions which aren't analytic, as follows.

Laurent's Theorem Suppose f is analytic in the annular domain $A = R_1 < |z - z_0| < R_2$ centered at z_0, and let C be a positive simple closed contour around z_0 lying in A. Then, for each $z \in A$, we have

$$\begin{aligned} f(z) &= \sum_{n\geq 0} a_n (z-z_0)^n + \sum_{n\geq 1} \frac{b_n}{(z-z_0)^n} \\ a_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz \\ b_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz \end{aligned}$$

Equivalently,

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \\ c_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz. \end{aligned}$$

If arbtirarily many coefficients c_N, $N < 0$, are nonzero, then the function is said to have an essential singularity. Otherwise, let N be the smallest integer such that $c_N \neq 0$; this is also denoted $\operatorname{ord}_{z_0}(f)$. If $N < 0$, then f has a pole of order $-N$ at z_0. If $N \geq 0$, then f has a zero of order N at z_0.

f has a zero of order N if $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ but $f^{(N)}(z_0) \neq 0$.

Definition A function f is called meromorphic if there is a discrete set $Z \subset \mathbb{C}$ such that $f|_{\mathbb{C}-Z}$ is analytic; Z is discrete; and for $z_0 \in Z$, f has a pole (of finite order) at z_0.

Proposition Let $S \subset \mathbb{C}$ be open and connected. Then the set of all meromorphic functions on S is a field.

Proof The only issue is quotients; but if an analytic function f vanishes on some set with a limit point, then it's actually identically zero. Similarly, if it has a zero of "infinite order", then it is identically zero. □

Suppose that f is analytic on and inside some positive, simple closed contour C which contains z_0, except at z_0. The residue of f at z_0 is

$$\begin{aligned} \operatorname{res}(f; z_0) = \operatorname{res}_{z=z_0} f(z) &= \frac{1}{2\pi i}\int_C f(z) \\ &= b_1 \\ &= c_{-1} \end{aligned}$$

where

$$\begin{aligned} f(z) &= \sum a_n (z-z_0)^n + \sum \frac{b_n}{(z-z_0)^n} \\ &= \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n \end{aligned}$$

is the Laurent series expansion.

Roughly, what's happening is: Let C be the unit circle around the origin. Then

$$\int_C z^n dz = \begin{cases} 0 & n \geq 0 \\ 2\pi i & n = -1 \\ 0 & n < -1 \end{cases}.$$

So, if you have a function with a Laurent series expansion, integrating around z_0 picks off the coefficient of $1/(z-z_0)$.

If you know how to calculate residues, then you know how to calculate zeros and poles:

Theorem If f is meromorphic, then

$$\operatorname{res}(\frac{f'}{f}; z_0) = \operatorname{ord}_{z_0}(f).$$

Proof If f has a zero, follows from a direct calculation. Now use the fact that $(fg)'/(fg) = (f'/f) + (g'/g)$, and the residue is additive. □

It is only the residues which contribute to a contour integral:

Residue theorem Suppose C is a simple positive closed contour, with interior D. Suppose f is meromorphic, and analytic on C. Then

$$\int_C f(z)dz = 2\pi i \sum_{P\in D} \operatorname{res}(f; P).$$

2 Review: The upper half-plane

The upper half plane is

$$\mathbb{H} = \mathbb{H}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

We sometimes also use the lower half-plane

$$\mathbb{H}^- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$$

so that $\mathbb{C}$ is a disjoint union

$$\mathbb{C} = \mathbb{H} \cup \mathbb{H}^- \cup \mathbb{R}.$$

Set $\mathbb{H}^\pm = \mathbb{H} \cup \mathbb{H}^-$.

In this section, we review some facts about automorphisms of $\mathbb{H}$.

Suppose $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. We define the function

$$\begin{aligned} \mathbb{C} &\xrightarrow{f_\alpha} \mathbb{C} \\ z &\longmapsto \frac{az+b}{cz+d}. \end{aligned}$$

Question 2.1. *a. Let $\alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Describe the effect of f_α on $\mathbb{C}$.*

b. Let $\beta = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Describe the effect of f_β on $\mathbb{C}$.

c. Let $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Describe the effect of f_γ on $\mathbb{C}$.

These functions satisfy some easy formal properties:

Question 2.2. *a. Show that f_α is meromorphic on $\mathbb{C}$.*

b. Suppose $\alpha \in \mathrm{GL}_2(\mathbb{C})$ and $\lambda \in \mathbb{C}^\times$. Show there is an equality of (meromorphic) functions

$$f_\alpha = f_{\lambda\alpha}.$$

c. Suppose $\alpha, \beta \in \mathrm{GL}_2(\mathbb{C})$. Show that there is an equality of functions

$$f_{\alpha\beta} = f_\alpha \circ f_\beta.$$

If we focus on matrices with real coefficients, we start getting at the structure of the upper half plane.

Question 2.3. *a. Suppose $\alpha \in \mathrm{GL}_2(\mathbb{R})$. Show that α is holomorphic on $\mathbb{H}^\pm$.*

b. *Suppose* $\alpha \in \mathrm{GL}_2(\mathbb{R})^+$*, i.e., that* $\det(\alpha) > 0$*. Show there exists some* $\beta \in \mathrm{SL}_2(\mathbb{R})$ *such that*

$$f_\alpha = f_\beta.$$

c. *In the previous question, is* β *unique?*

It turns out that what we actually want to study is those functions which preserve $\mathbb{H}$.

Question 2.4. *Suppose* $\alpha \in \mathrm{SL}_2(\mathbb{R})$*. Show that* $f_\alpha(\mathbb{H}) \subseteq \mathbb{H}$*.*

Here are two different ways of doing this:

a. *Directly compute* $\mathrm{Im}(f_\alpha(z))$*. (Remember,* $\mathrm{Im}(w) = \frac{1}{2i}(w - \overline{w})$*.)*

b. *This is (much?) more work, but maybe more fun:*

   (i) *Show that* $0 \notin f_\alpha(\mathbb{H}^\pm)$*.*

   (ii) *Show that* $f_\alpha(\mathbb{H}^\pm)$ *contains no element of* $\mathbb{R}$*. (*HINT: Use problem 2.1.*)*

   (iii) *Show that* $\mathrm{SL}_2(\mathbb{R})$ *is connected (either use the Iwasawa decomposition, or point-counting over finite fields and Lang-Weil).*

   (iv) *Show that there exists some* $\beta \in \mathrm{SL}_2(\mathbb{R})$ *and some* $z_0 \in \mathbb{H}$ *such that* $f_\beta(z_0) \in \mathbb{H}$*.*

   (v) *Explain why this shows that, for each* $\alpha \in \mathrm{SL}_2(\mathbb{R})$ *and each* $z \in \mathbb{H}$*,* $f_\alpha(z) \in \mathbb{H}$*. (*HINT: Use connectedness.*)*

Moreover, we can use $\mathrm{SL}_2(\mathbb{R})$ to give a construction of $\mathbb{H}$ as a quotient space:

Question 2.5. a. *Show that* $\mathrm{SL}_2(\mathbb{R})$ *acts transitively on* $\mathbb{H}$*. (*HINT: Given $z_0 \in \mathbb{H}$, first show how to find an $\alpha \in \mathrm{SL}_2(\mathbb{R})$ such that $\mathrm{Re}(f_\alpha(z_0)) = 0$. Now find a β such that $f_\beta(f_\alpha(z_0)) = i$.*)*

b. *Describe the stabilizer*

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) := \{\alpha \in \mathrm{SL}_2(\mathbb{R}) : f_\alpha(i) = i\}.$$