Homework 6

Due: Friday, November 4

In the first two problems, we'll work with the power maps $p_d : \mathbb{G}_m \to \mathbb{G}_m$, *the covering map* $g : \mathbb{G}_m \to \mathbb{G}_m$ *given by* $z \mapsto z + z^{-1}$, *and the Chebychev maps* $T_d : \mathbb{C} \to \mathbb{C}$. *Fix* $d \geq 2$.

1. *Fixed points of* T_d
   (a) Compute the fixed points of p_d.
   (b) Use this to compute the fixed points of T_d. (HINT: *You can express them using the cosine function; if* $t \in \mathbb{R}$, *what is* $g(\exp(it))$*?*)
2. *Multipliers of* T_d For the fixed points ζ of T_d you found in the previous problem, compute the multipler $\lambda_\zeta(T_d) = T_d'(\zeta)$. (HINT: *Differentiate the relation*
$$T_d(z + z^{-1}) = z^d + z^{-d}$$
   to find an expression for $T_d'(z + z^{-1})$.)
   In fact, one can show that $\sum_\zeta \frac{1}{1-\lambda_\zeta(T_d)} = 1$.
3. *Endomorphisms of elliptic curves* Let $\Lambda \subset \mathbb{C}$ be a lattice. Suppose that $\alpha \in \mathbb{C}$ satisfies $\alpha\Lambda \subseteq \Lambda$.
   (a) Show that α is actually an algebraic integer, of degree at most 2. (In other words, show that there are integers p and q such that $\alpha^2 + p\alpha + q = 0$.) (HINT: *Choose a basis* $\{\omega_1, \omega_2\}$ *for* Λ, *and think of* α *as a linear transformation from* Λ *to itself. What can you say about its characteristic polynomial?*)
   (b) Suppose $\alpha \notin \mathbb{Z}$. Show that α is an imaginary quadratic integer.
4. *Multiplication by* 2 *on elliptic curves* Suppose E is given by a equation $y^2 = x^3 + ax + b$, and $P = (x, y) \in E(K)$ is not $\mathcal{O}$, $y = y(P) \neq 0$. Find a formula for $2P$:
   (a) What is the slope of the tangent line L to E at P?
   (b) Find a formula for L.
   (c) Find all points of intersection of L and E.

   You should be able to get a formula for $2P$ *of the form* $(g(x, y), h(x, y))$, *where* g *and* h *are rational functions. If the denominator of* g *vanishes for some particular point* $P_0 = (x_0, y_0)$, *we interpret this as* $2P_0 = \mathcal{O}$.