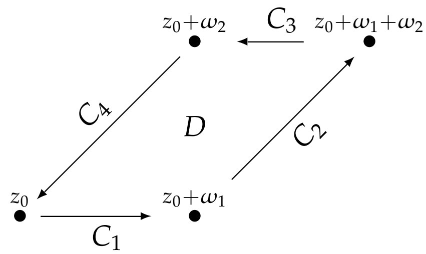
Homework 5

Due: Monday, October 17

In these problems, we work with a lattice $\Lambda \subset \mathbb{C}$ generated by ω_1 and ω_2.

Let D be a fundamental domain for Λ, and $C = C_1 + C_2 + C_3 + C_4$ a contour around D, as below.



(1)

1. Suppose f is meromophic and Λ-periodic. Show that f' is, too.
2. Suppose f is meromorphic, Λ-periodic, and has no poles or zeros along C. Show that $\int_C f(z)\,dz = 0$.
3. Let h be a meromorphic function, and suppose $z_0 \in \mathbb{C}$, $z_0 \neq 0$. Show that

$$\mathrm{res}_{z_0}\left(\frac{zh'(z)}{h(z)}\right) = \mathrm{ord}_{z_0}(h) \cdot z_0.$$

4. Suppose f is meromorphic, Λ-periodic, and has no poles or zeros along C. Show that

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz \in \Lambda. \tag{2}$$

Note that the integrand $zf'(z)/f(z)$ is *not* periodic.

You are welcome to take the following steps. You may want to use the fact that log is actually a multifunction, so that if C is a contour from α to β, and if g is any continuous, nonvanishing function on C, then

$$\int_C \frac{g'(z)}{g(z)}dz = \log(g(z))\Big|_\alpha^\beta \in (\mathrm{Log}(\beta) - \mathrm{Log}(\alpha)) + 2\pi i\mathbb{Z}.$$

You will probably have to explicitly parametrize the contours C_j in order to evaluate the integrals.

(a) Show that $\int_{C_3} \frac{zf'(z)}{f(z)}dz = \int_{-C_1} \frac{zf'(z+\omega_2)}{f(z+\omega_2)}dz + \omega_2 \int_{C_3} \frac{f'(z)}{f(z)}dz$. (Hint: $zf'(z)/f(z) = (z-\omega_2)f'(z)/f(z) + \omega_2 f'(z)/f(z)$.)

(b) Show that $\int_{C_1+C_3} \frac{zf'(z)}{f(z)}dz \in 2\pi i\omega_2\mathbb{Z}$.

(c) Show (2).

5. Suppose f is meromorphic, Λ-periodic, and has no zeros or poles along C.

   Suppose the only zeros of f in D are $P_1, \cdots, P_r$, and that f has a zero of order m_i at P_i. Similarly, suppose the only poles of f in D are $Q_1, \cdots, Q_s$, and that f has a pole of order n_j at Q_j. Show that

$$\sum_{i=1}^{r} m_i P_i \equiv \sum_{j=1}^{s} n_j Q_j \bmod \Lambda.$$