Homework 3

Due: Monday, September 26

[M] refers to Milnor's Dynamics in one complex variable, *third edition. Where appropriate, I will also indicate, using [M_1], corresponding problem in the first edition of Milnor's notes, which is available on the* `arXiv`

1. (a) [M] 1-d. (See also [M_1]1-10)

   (b) [M] 1-e. ([M_1]1-12)

2. Let $\Gamma \subset \mathbb{R}^n$ be an (additive) group of rank r. Thus, there are $\gamma_1, \cdots, \gamma_r \in \mathbb{Z}$ which are $\mathbb{Z}$-linearly independent such that

$$\Gamma = \{\sum a_i \gamma_i : a_i \in \mathbb{Z}\}.$$

   Show that Γ is discrete $\iff$ $\{\gamma_1, \cdots, \gamma_r\}$ is $\mathbb{R}$-linearly independent.

3. Recall the Cayley transform

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{F} & \mathbb{D} \\ z & \longmapsto & \dfrac{z-i}{z+i} \end{array}$$

   Consider the Poincaré metric

$$ds = 2\frac{|dz|}{1-|z|^2}$$

   on $\mathbb{D}$. Compute its pullback F^*ds to $\mathbb{H}$.

4. A Riemannian metric can be used to define the length of a (parametrized) curve; if $\Omega \subset \mathbb{C}$ is a Riemann surface with metric $ds = \rho(z)|dz|$, and if $\gamma : [a, b] \to \Omega$ is a curve, then the length of γ is

$$\ell(\gamma) = \ell_\rho(\gamma) = \int_a^b \rho(\gamma(t))\left|\gamma'(t)\right| dt.$$

   Conside $\mathbb{D}$ with the Poincaré metric.

   (a) Fix $0 < b < 1$, and consider the arc

$$\begin{array}{ccc} [0, b] & \xrightarrow{\gamma} & \mathbb{D} \\ t & \longmapsto & t \end{array}$$

Show that

$$\ell(\gamma) = \log \frac{1+b}{1-b}.$$

In fact, $\operatorname{dist}(0, b) = \log \frac{1+b}{1-b}$.

(b) Suppose $z_0 \in \mathbb{D}$. Show that

$$\operatorname{dist}(0, z_0) = \log \frac{1+|z|}{1-|z|}.$$

(HINT: *A rotation is an isometry*)

(c) Suppose $z_0, z_1 \in \mathbb{D}$. Show that

$$\operatorname{dist}(z_0, z_1) = \log \frac{1 + \left|\frac{z_0 - z_1}{1 - \bar{z}_0 z_1}\right|}{1 - \left|\frac{z_0 - z_1}{1 - \bar{z}_0 z_1}\right|}.$$

(HINT: *A Blaschke factor is an isometry.*)

5. [M] 2-e. *Feel free to make use of earlier problems in [M].*