Homework 3 Due: Monday, September 26

[M] refers to Milnor's Dynamics in one complex variable, third edition. Where appropriate, I will also indicate, using $[M_1]$, corresponding problem in the first edition of Milnor's notes, which is available on the arXiv

- 1. (a) [M] 1-d. (See also [M₁]1-10)
 - (b) [M] 1-e. ([M₁]1-12)
- 2. Let $\Gamma \subset \mathbb{R}^n$ be an (additive) group of rank r. Thus, there are $\gamma_1, \dots, \gamma_r \in \mathbb{Z}$ which are \mathbb{Z} -linearly independent such that

$$\Gamma = \{\sum a_i \gamma_i : a_i \in \mathbb{Z}\}.$$

Show that Γ is discrete $\iff \{\gamma_1, \cdots, \gamma_r\}$ is \mathbb{R} -linearly independent.

3. Recall the Cayley transform

$$\mathbb{H} \xrightarrow{F} \mathbb{D}$$
$$z \longmapsto \frac{z - i}{z + i}$$

Consider the Poincaré metric

$$ds = 2\frac{\left|dz\right|}{1 - \left|z\right|^2}$$

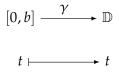
on \mathbb{D} . Compute its pullback F^*ds to \mathbb{H} .

4. A Riemannian metric can be used to define the length of a (parametrized) curve; if $\Omega \subset \mathbb{C}$ is a Riemann surface with metric $ds = \rho(z)|dz|$, and if $\gamma : [a, b] \to \Omega$ is a curve, then the length of γ is

$$\ell(\gamma) = \ell_{\rho}(\gamma) = \int_{a}^{b} \rho(\gamma(t)) |\gamma'(t)| dt.$$

Conside $\ensuremath{\mathbb{D}}$ with the Poincaré metric.

(a) Fix 0 < b < 1, and consider the arc



Professor Jeff Achter Colorado State University Math 619: Complex analysis II Fall 2011 Show that

$$\ell(\gamma) = \log \frac{1+b}{1-b}.$$

In fact, dist $(0, b) = \log \frac{1+b}{1-b}$.

(b) Suppose $z_0 \in \mathbb{D}$. Show that

dist
$$(0, z_0) = \log \frac{1 + |z|}{1 - |z|}.$$

(HINT: A rotation is an isometry)

(c) Suppose $z_0, z_1 \in \mathbb{D}$. Show that

dist
$$(z_0, z_1) = \log \frac{1 + \left| \frac{z_0 - z_1}{1 - \overline{z}_0 z_1} \right|}{1 - \left| \frac{z_0 - z_1}{1 - \overline{z}_0 z_1} \right|}.$$

(HINT: A Blaschke factor is an isometry.)

5. [M] 2-e. Feel free to make use of earlier problems in [M].