Homework 2

Due: Friday, September 9

1. Consider the unit sphere $X = \{(a,b,c) : a^2+b^2+c^2 = 1\} \subset \mathbb{R}^3$. Let $N = (0,0,1)$, $S = (0,0,-1)$, $U_N = X - \{N\}$, $U_S = X - \{S\}$. Consider the following three charts on X:

$$U_N \xrightarrow{\phi_N} \mathbb{C}$$

$$(a_0,b_0,c_0) \longmapsto \frac{a_0+ib_0}{1-c_0}$$

$$U_S \xrightarrow{\phi_S} \mathbb{C}$$

$$(a_0,b_0,c_0) \longmapsto \frac{a_0+ib_0}{1+c_0}$$

$$U_S \xrightarrow{\psi_S} \mathbb{C}$$

$$(a_0,b_0,c_0) \longmapsto \frac{a_0-ib_0}{1+c_0}$$

   (a) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = (\frac{2\Re(z)}{|z|^2+1}, \frac{2\Im(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}).$$

   Calculate $\phi_S^{-1}(z)$ and $\psi_S^{-1}(z)$.

   (b) Among the three charts $\{(U_N,\phi_N), (U_S,\phi_S), (U_S,\psi_S)\}$, one pair is compatible and the other two are not. Which is which? Why? (HINT: *Remember that a function f is holomorphic if and only if $\partial_{\bar{z}} f = 0$; colloquially, a function is holomorphic if it doesn't involve any $\bar{z}$'s.*)

2. As a set, the complex projective line $\mathbb{P}^1$ is defined as follows. Delete the origin from $\mathbb{C}^2$, and then define an equivalence relation on the remaining set, by saying that $(a_0,a_1) \sim (b_0,b_1)$ if and only if there exists some $\lambda \in \mathbb{C}^\times$ so that $b_0 = \lambda a_0$ and $b_1 = \lambda a_1$.

   Thus,

$$\mathbb{P}^1 = (\mathbb{C}^2 - \{0\})/\sim,$$

   and the equivalence class of (a_0,a_1) in $\mathbb{P}^1$ is denoted $[a_0,a_1]$.

   Consider the map

$$\mathbb{P}^1 \xrightarrow{\alpha} \mathbb{R}^3$$

$$[z_0,z_1] \longmapsto (\frac{2\Re(z_1\overline{z_0})}{|z_1|^2+|z_0|^2}, \frac{2\Im(z_1\overline{z_0})}{|z_1|^2+|z_0|^2}, \frac{|z_1|^2-|z_0|^2}{|z_1|^2+|z_0|^2}).$$

(a) Show that this really is a function on $\mathbb{P}^1$, i.e., if $\lambda \in \mathbb{C}^\times$, then $\alpha([\lambda z_0, \lambda z_1]) = \alpha([z_0, z_1])$.

(b) Show that the image of α is the unit sphere $a^2 + b^2 + c^2 = 1$. (In fact, α is a homeomorphism.) (HINT: *Remember that for any* $w \in \mathbb{C}$, $\Re(w) = \frac{w+\overline{w}}{2}$, $\Im(w) = \frac{w-\overline{w}}{2i}$, *and* $|w|^2 = w\overline{w}$.)

3. For $j \in \{0,1\}$, let $V_j = \{[z_0, z_1] : z_j \neq 0\} \subset \mathbb{P}^1$; and define the charts (V_j, ϕ_j) by

$$V_j \xrightarrow{\phi_j} \mathbb{C}$$

$$[z_0, z_1] \longmapsto \frac{z_{1-j}}{z_j}.$$

Endow the unit sphere X with the compatible charts you found in problem (1). Show that the map α from problem (2) is holomorphic.