Homework 1

Due: Friday, September 2

1. Let $p(z)$ be a nonconstant polynomial. Show that the Julia set $J(z)$ is bounded.

2. Let $p(z)$ be a polynomial of degree at least two. Show that $J(p)$ is nonempty:

   Suppose the Fatou set $F(p)$ is all of $\mathbb{C}$, so that there is an increasing sequence $\{n_j\}$ such that $\{p^{\circ n_j}\}$ converges uniformly on $\mathbb{C}$. (This is a little hard to show given our foundations, so just assume this is implied by the assumption $F(p) = \mathbb{C}$.)

   (a) Show that there is some index k and a compact set $K \subset \mathbb{C}$ such that if $z \in \mathbb{C} - K$ and $j > k$, then $p^{\circ n_j}(z) \neq 0$.

   (b) Show that for $j > k$, the number of zeros of $p^{\circ n_j}$, counted with multiplicity, is same as the number of zeros of $p^{\circ n_k}$. (HINT: *Use the argument principle and a circular contour whose interior contains K.*)

   (c) Derive a contradiction, and thus conclude $J(p) \neq \emptyset$.

3. Consider the polynomial $p(z) = z^2 - 2$. Let I be the real interval $I = [-2, 2] \subset \mathbb{C}$.

   (a) Show that $p(I) = I$.

   (b) Suppose $x \in \mathbb{R} - I \subset \mathbb{C}$. Show that

   $$\lim_{n \to \infty} |p^{\circ n}(x)| = \infty.$$

   (c) *[Possibly much harder]* Suppose $z \in \mathbb{C} - I$. Show that

   $$\lim_{n \to \infty} |p^{\circ n}(z)| = \infty.$$