Homework 9
Due(*): Friday, October 19

(*) You only have to hand in the last problem. Use this time to think about the interconnections between line bundles, sections and functions. Please stop by if you have questions!

1. Let $X = V / \Lambda$ be an elliptic curve. Suppose that $\mathcal{L}$ is a line bundle on $X$ corresponding to the following action of $\Lambda$ on $\mathcal{L}_{\text{triv}} = V \times \mathbb{C}$:

$$\Lambda \times (V \times \mathbb{C}) \rightarrow V \times \mathbb{C}$$

$$\lambda \times (z, t) \mapsto (z + \lambda, \beta_{\lambda}(z) \cdot t)$$

Similarly, suppose $\mathcal{M}$ corresponds to $(z, t) \mapsto (z + \lambda, \gamma_{\lambda}(z)) \cdot t$.

(a) What action corresponds to the tensor product $\mathcal{L} \otimes \mathcal{M}$?

(b) Suppose $s$ is a section of $\mathcal{L}$ and $t$ is a section of $\mathcal{M}$. Convince yourself that $s \cdot t$ is a section of $\mathcal{L} \otimes \mathcal{M}$. What sort of transformation rule does the function $\theta_{s,t}$ satisfy?

(c) Relate this to Homework 6, problem 3.

2. Fix a lattice $\Lambda = \Lambda_{\tau}$. Fix a natural number $\ell \geq 2$. Let $T = \left((\frac{1}{\ell} \mathbb{Z})/\mathbb{Z}\right) \times \left((\frac{1}{\ell} \mathbb{Z})/\mathbb{Z}\right)$. For each $(a, b) \in T$, define

$$\theta_{a,b}^*(z) = \theta\left[\begin{array}{c} a \\ b \end{array}\right](\ell z).$$

(a) Show that the functions $\{\theta_{a,b}^*(z) : (a, b) \in T\}$ are theta functions of the same type with respect to $\Lambda$. Therefore, they are all sections of the same line bundle $\mathcal{L}_\ell$ on the elliptic curve $X = V / \Lambda$.

(b) In class, we saw that if $\mathcal{M}$ is a line bundle on complex manifold $Y$, then a collection of sections of $\mathcal{M}$ defines a map from $Y$ into some projective space. Convince yourself that the embedding $\alpha : X \hookrightarrow \mathbb{P}^3$ arises in this way.

3. Let $V = \mathbb{C}^2$ be a two-dimensional complex vector space

A lattice $\Lambda \subset V$ is not simple if there is a decomposition $V = W_1 \oplus W_2$, a sum of one-dimensional complex subspaces, such that

$$\Lambda = (\Lambda \cap W_1) \oplus (\Lambda \cap W_2).$$

Show that there are lattices which are simple.

(HINT: Briefly, think of $V$ as a vector space over $\mathbb{R}$. If $W \subset V$ is a real subspace, then $W$ is also a complex subspace if and only if it is stable under multiplication by $i$. Compare Debarre exercise 1.2.)

Extra credit: Let $X = \mathbb{C}/\Lambda$ be an elliptic curve. There is a line bundle $\mathcal{L}$ on $X$, and sections $s_0, s_1, s_2$ of $\mathcal{L}$, which give rise to the Weierstrass embedding $X \hookrightarrow \mathbb{P}^2$. Which line bundle is it? Explain. (Hint: See Debarre, Proposition 2.12.)