Homework 9

Due$^{(*)}$: Friday, October 19

(*) *You only have to hand in the last problem. Use this time to think about the interconnections between line bundles, sections and functions. Please stop by if you have questions!*

1. Let $X = V/\Lambda$ be an elliptic curve. Suppose that $\mathcal{L}$ is a line bundle on X corresponding to the following action of Λ on $\mathcal{L}_{\text{triv}} = V \times \mathbb{C}$:

$$\Lambda \times (V \times \mathbb{C}) \longrightarrow V \times \mathbb{C}$$

$$\lambda \times (z, t) \longmapsto (z + \lambda, \beta_\lambda(z) \cdot t)$$

   Similarly, suppose $\mathcal{M}$ corresponds to $(z, t) \mapsto (z + \lambda, \gamma_\lambda(z)) \cdot t$.

   (a) What action corresponds to the tensor product $\mathcal{L} \otimes \mathcal{M}$?

   (b) Suppose s is a section of $\mathcal{L}$ and t is a section of $\mathcal{M}$. Convince yourself that $s \cdot t$ is a section of $\mathcal{L} \otimes \mathcal{M}$. What sort of transformation rule does the function $\theta_{s \cdot t}$ satisfy?

   (c) Relate this to Homework 6, problem 3.

2. Fix a lattice $\Lambda = \Lambda_\tau$. Fix a natural number $\ell \geq 2$. Let $T = ((\frac{1}{\ell}\mathbb{Z})/\mathbb{Z}) \times ((\frac{1}{\ell}\mathbb{Z})/\mathbb{Z})$. For each $(a, b) \in T$, define

$$\theta^*_{a,b}(z) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\ell z).$$

   (a) Show that the functions $\{\theta^*_{a,b}(z) : (a, b) \in T\}$ are theta functions of the same type with respect to Λ. *Therefore, they are all sections of the same line bundle $\mathcal{L}_\ell$ on the elliptic curve $X = V/\Lambda$.*

   (b) In class, we saw that if $\mathcal{M}$ is a line bundle on complex manifold Y, then a collection of sections of $\mathcal{M}$ defines a map from Y into some projective space. Convince yourself that the embedding $\alpha : X \hookrightarrow \mathbb{P}^3$ arises in this way.

3. Let $V = \mathbb{C}^2$ be a two-dimensional complex vector space

   A lattice $\Lambda \subset V$ is *not simple* if there is a decomposition $V = W_1 \oplus W_2$, a sum of one-dimensional complex subspaces, such that

$$\Lambda = (\Lambda \cap W_1) \oplus (\Lambda \cap W_2).$$

   Show that there are lattices which *are* simple.

   (HINT: *Briefly, think of V as a vector space over $\mathbb{R}$. If $W \subset V$ is a real subspace, then W is also a complex subspace if and only if it is stable under multiplication by i. Compare Debarre exercise 1.2.*)

Extra credit: Let $X = \mathbb{C}/\Lambda$ be an elliptic curve. There is a line bundle $\mathcal{L}$ on X, and sections s_0, s_1, s_2 of $\mathcal{L}$, which give rise to the Weierstrass embedding $X \hookrightarrow \mathbb{P}^2$. Which line bundle is it? Explain. (Hint: See Debarre, Proposition 2.12).