Homework 8

Due: Friday, October 12

In contravention of all sense and custom, I've been writing transition maps backwards. Typically, one arranges things so that $g_{\alpha\beta}$ is the transition function from U_β to U_α, which unfortunately is the opposite of how I've been writing it. In this problem set, I will explicitly write things like $g_{U_\beta \leftarrow U_\alpha}$ to move from U_α-coordinates to U_β-coordinates. In particular, if a line bundle $\mathcal{L}$ corresponds to data $\{U_\alpha, g_{U_\beta \leftarrow U_\alpha}\}$, then a section s of $\mathcal{L}$ corresponds to holomorphic functions $s_\alpha \in \mathcal{H}(U_\alpha)$ such that, on $U_\alpha \cap U_\beta$, we have

$$s_\beta = g_{U_\beta \leftarrow U_\alpha} \cdot s_\alpha.$$

1. $\boxed{\mathcal{O}_{\mathbb{P}^n}(-1)}$ Recall that we have identified each point $P = [a_0, \cdots, a_n] \in \mathbb{P}^n$ with a line L_P in $\mathbb{C}^{n+1}$; then $\mathcal{O}_{\mathbb{P}^n}(-1)$, as a set, is

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(P, s) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : s \in L_P\}.$$

Define trivializations on the standard open patches U_i:

$$\mathcal{O}_{\mathbb{P}^n}(-1)|_{U_i} \xrightarrow{\psi_i} U_i \times \mathbb{C}$$

$$(P, s) \longmapsto (P, s_i)$$

$$(P, \frac{\lambda}{a_i} P) \longleftarrow (P, \lambda)$$

where $\frac{\lambda}{a_i} P = (\lambda \frac{a_0}{a_i}, \cdots, \lambda \frac{a_n}{a_i})$. (Note that this is well-defined!)

Show that the transition maps are

$$g_{U_j \leftarrow U_i}([a_0, \cdots, a_n]) = \frac{a_j}{a_i}.$$

2. $\boxed{\mathcal{O}_{\mathbb{P}^n}(1)}$ If $\mathcal{L} \leftrightsquigarrow \{U_\alpha, g_{U_\beta \leftarrow U_\alpha}\}$, its dual $\mathcal{L}^\vee$ is the line bundle with data $\{U_\alpha, \frac{1}{g_{U_\beta \leftarrow U_\alpha}}\}$.

   (a) Describe the transition functions for $\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}_{\mathbb{P}^n}(-1)^\vee$.

   (b) Fix numbers $b_0, \cdots, b_n$. For each $0 \le i \le n$, define a holomorphic function on U_i:

$$L_i([a_0, \cdots, a_n]) = \sum_k b_k \frac{a_k}{a_i}.$$

   Show that the data $\{L_i : 0 \le i \le n\}$ defines a section of $\mathcal{O}_{\mathbb{P}^n}(1)$.

In fact: sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ are in bijection with linear homogeneous polynomials in $X_0, \cdots, X_n$.

3. $\boxed{\mathcal{O}_{\mathbb{P}^n}(2)}$ Let $\mathcal{O}_{\mathbb{P}^n}(2) = \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$.

   (a) Use the results of the previous question to describe some sections of $\mathcal{O}_{\mathbb{P}^n}(2)$.

   (b) If $r \geq 1$, let $\mathcal{O}_{\mathbb{P}^n}(r) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes r}$. Give a (conjectural) description of the sections of $\mathcal{O}_{\mathbb{P}^n}(r)$.

4. *For use later:*

   (a) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. *In fact, the two copies of $\mathbb{C}$ on the right-hand side can be indexed by the two $\mathbb{R}$-linear isomorphisms $\mathbb{C} \to \mathbb{C}$.*

   (b) Suppose V is a complex vector space. Since there's a unique inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$, V is also a real vector space. Describe the $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$-module $V \otimes_{\mathbb{R}} \mathbb{C}$.