Homework 7
Due: Friday, October 5

Fix a lattice $\Lambda = \Lambda_\tau$. Theta functions can be used to construct meromorphic functions on $E = \mathbb{C}/\Lambda$ in a few different ways:

1. Suppose θ_1 and θ_2 have the same type. Show that the function $\frac{\theta_1}{\theta_2}$ is Λ-periodic, so that it defines a function on E.
2. Let θ be a theta function. Fix numbers $P_1, \cdots, P_n, Q_1, \cdots, Q_n \in \mathbb{C}$ such that $\sum P_j = \sum Q_j$, and $\{P_1, \cdots, P_n\} \cap \{Q_1, \cdots, Q_n\} = \emptyset$. Consider the function
$$f(z) = \frac{\theta(z-P_1)\theta(z-P_2)\cdots\theta(z-P_n)}{\theta(z-Q_1)\theta(z-Q_2)\cdots\theta(z-Q_n)}.$$
   (a) Show that f is Λ-periodic.
   (b) Describe the zeros and poles of f.
   (c) Is your work in (b) compatible with HW3#4? Explain.
3. Consider the function
$$g(z) = \frac{\partial^2}{\partial z^2} \log \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z).$$
   (a) Describe all poles of $g(z)$.
   (b) By HW6#4, $g(z)$ is Λ-periodic. Therefore, $g(z) \in \mathbb{C}(\wp(z), \wp'(z))$. Which function is it? *Say as much as you can, even if you can't answer this exactly.*
4. In class, we found that for any $x \in \mathbb{C}$,
$$\theta_{00}(0)^2\theta_{00}(x)^2 = \theta_{01}(0)^2\theta_{01}(x)^2 + \theta_{10}(0)^2\theta_{10}(x)^2$$
$$\theta_{10}(0)^2\theta_{01}(x)^2 - \theta_{01}(0)^2\theta_{10}(x)^2 + \theta_{00}(0)^2\theta_{11}(x)^2 = 2\theta_{00}(0)^2\theta_{11}(x)^2$$
   The second equation can of course be replaced by
$$\theta_{00}(0)^2\theta_{11}(x)^2 = \theta_{10}(0)^2\theta_{01}(x)^2 - \theta_{01}(0)^2\theta_{10}(x)^2$$
   So, let $W_0, \cdots, W_3$ be coordinates on $\mathbb{P}^3$. This shows that the image of
$$\mathbb{C}/\Lambda \xrightarrow{\alpha = \alpha_{2,\tau}} \mathbb{P}^3$$
$$x \longmapsto [\theta_{00}(2x), \theta_{01}(2x), \theta_{10}(2x), \theta_{11}(2x)]$$

lies in the vanishing locus of the (homogeneous!) polynomials

$$F = \theta_{00}(0)^2 W_0^2 - (\theta_{01}(0)^2 W_1^2 + \theta_{10}(0)^2 W_2^2)$$
$$G = \theta_{00}(0)^2 W_3^2 - (\theta_{10}(0)^2 W_1^2 - \theta_{01}(0)^2 W_2^2)$$

Let $Y \subset \mathbb{P}^3$ be the vanishing locus of F and G. In this problem we'll show that α is surjective. Recall that a hyperplane in $\mathbb{P}^3$ is the vanishing locus of some linear polynomial $\sum a_i W_i$, where not all a_i are zero.

(a) If you've taken 672, show that if H is a hyperplane, then $\# H \cap Y \leq 4$. *If you haven't, just accept this.*

(b) Fix numbers $a_0, \cdots, a_3$, not all zero. Consider the function

$$f(x) = a_0\theta_{00}(x) + a_1\theta_{01}(x) + a_2\theta_{10}(x) + a_3\theta_{10}(x).$$

Show that f has exactly four zeros in any fundamental domain for 2Λ.

(c) Show that $\alpha : \mathbb{C}/\Lambda \to Y$ is surjective.