Homework 5
Due: Friday, September 21

Throughout this problem set, let $F(Z_0, \cdots, Z_n) \in \mathbb{C}[Z_0, \cdots, Z_n]$ be homogeneous of degree d. Let $W = \{P \in \mathbb{P}^n : F(P) = 0\} \subset \mathbb{P}^n$ be the vanishing locus of F.

1. *Dehomogenization* Let $f(x_1, \cdots, x_n) = F(1, x_1, \cdots x_n)$.

   (a) Show that $Z_0^d f(Z_1/Z_0, \cdots, Z_n/Z_0) = F(Z_0, \cdots, Z_n)$.
   In fact, the affine coordinates x_j *are related to the* projective coordinates Z_j *by* $x_j = Z_j/Z_0$.

   (b) Suppose $P = [a_0, \cdots, a_n] \in U_0$. Show that $F(P) = 0$ if and only if $f(a_1/a_0, \cdots, a_n/a_0) = 0$.
   In fact, let $\phi_0 : U_0 \to \mathbb{C}^n$ be the usual chart. Then $\phi_0(W \cap U_0)$ is the vanishing locus of f in $\mathbb{C}^n$.

2. *A formula of Euler* Let $F(Z_0, \cdots, Z_n) \in \mathbb{C}[Z_0, \cdots, Z_n]$ be homogeneous of degree d. Prove Euler's formula:
$$F = \frac{1}{d}\sum_{j=0}^{n} Z_j \cdot \frac{\partial}{\partial Z_j} F.$$
   (HINT: *Prove this for monomials.*)

3. Let $F \in \mathbb{C}[Z_0, \cdots, Z_n]$ be homogeneous of degree d.

   (a) Suppose $P \in \mathbb{P}^n$ is a point such that for each $j \in \{0, \cdots, n\}$, $\frac{\partial}{\partial Z_j} F|_P = 0$. Show that $F(P) = 0$, too.

   (b) Suppose that the partial derivatives $\frac{\partial}{\partial Z_j} F$ have no common zero. Show that W admits a structure of complex manifold.
   (HINT: *Consider the intersections $X \cap U_j$ of X with the various affine patches U_j. If $f(x_1, \cdots, x_n) = F(1, x_1, \cdots, x_n)$, then*
$$\frac{\partial}{\partial x_j} f|_{(a_1, \cdots, a_n)} = \frac{\partial}{\partial Z_j} F|_{[1, a_1, \cdots, a_n]}.$$
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4. Let $\Lambda \subset \mathbb{C}$ be a lattice. On Monday we will define its discriminant as
$$\Delta(\Lambda) = (60G_4(\Lambda))^3 - 27(140G_6(\Lambda))^2.$$
   and show that it is nonzero.

   (a) Explain how to define a Δ function on the upper half plane $\mathfrak{h}$ such that $\Delta(\tau) = \Delta(\Lambda_{1,\tau})$.

(b) Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. What is the relationship between $\Delta(M\tau)$ and $\Delta(\tau)$?

(c) Construct a (nonconstant) function f on $\mathfrak{h}$ of the form $f(\tau) = \frac{g(\tau)}{\Delta(\tau)}$ such that for all $\tau \in \mathfrak{h}$ and $M \in \mathrm{SL}_2(\mathbb{Z})$, $f(M\tau) = f(\tau)$.

(HINT: *See HW 4 #1.*)