Homework 4

Due: Friday, September 14

1. For a lattice Λ and a natural number $k \geq 3$, we have defined a constant

$$G_k = G_k(\Lambda) = \sum_{\lambda \in \Lambda'} \frac{1}{\lambda^k}.$$

Since any lattice Λ is homothetic to some Λ_τ, we study the function defined on the upper half-plane $\mathfrak{h}$:

$$G_k(\tau) = \sum_{m,n \in \mathbb{Z}: m,n \text{ not both } 0} \frac{1}{(m\tau + n)^k}.$$

   (a) Suppose that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Show that

$$G_k(M \cdot \tau) = (c\tau + d)^k G_k(\tau).$$

   (b) Use this to create some meromorphic functions which are defined on the quotient space $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$.[1]

2. Consider the unit sphere $X = \{(a, b, c) : a^2 + b^2 + c^2 = 1\} \subset \mathbb{R}^3$. Let $N = (0, 0, 1)$, $S = (0, 0, -1)$, $U_N = X - \{N\}$, $U_S = X - \{S\}$. Consider the following three charts on X:

$$U_N \xrightarrow{\phi_N} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 + ib_0}{1 - c_0}$$

$$U_S \xrightarrow{\phi_S} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 + ib_0}{1 + c_0}$$

$$U_S \xrightarrow{\psi_S} \mathbb{C}$$

$$(a_0, b_0, c_0) \longmapsto \frac{a_0 - ib_0}{1 + c_0}$$

   (a) The inverse of ϕ_N is

$$\phi_N^{-1}(z) = \left(\frac{2\Re(z)}{|z|^2 + 1}, \frac{2\Im(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

   Calculate $\phi_S^{-1}(z)$ and $\psi_S^{-1}(z)$.

[1] Really, explain how to create functions which are analytic on some open subset of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$.

(b) Among the three charts $\{(U_N, \phi_N), (U_S, \phi_S), (U_S, \psi_S)\}$, one pair is compatible and the other two are not. Which is which? Why? (HINT: *Remember that a function f is holomorphic if and only if $\partial_{\overline{z}} f = 0$; colloquially, a function is holomorphic if it doesn't involve any $\overline{z}$'s.*)

3. In class, on Monday, we will define projective n-space $\mathbb{P}^n$, and a complex structure on it. Consider the map

$$\mathbb{P}^1 \xrightarrow{\quad\alpha\quad} \mathbb{R}^3$$

$$[z_0, z_1] \longmapsto \left(\frac{2\Re(z_1\overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{2\Im(z_1\overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{|z_1|^2 - |z_0|^2}{|z_1|^2 + |z_0|^2}\right.$$

(a) Show that this really is a function on $\mathbb{P}^1$, i.e., if $\lambda \in \mathbb{C}^\times$, then $\alpha([\lambda z_0, \lambda z_1]) = \alpha([z_0, z_1])$.

(b) Show that the image of α is the unit sphere $a^2 + b^2 + c^2 = 1$. (In fact, α is a homeomorphism.) (HINT: *Remember that for any $w \in \mathbb{C}$, $\Re(w) = \frac{w+\overline{w}}{2}$, $\Im(w) = \frac{w-\overline{w}}{2i}$, and $|w|^2 = w\overline{w}$.*)

Extra credit: For $j \in \{0, 1\}$, we will let $U_j = \{[z_0, z_1] : z_j \neq 0\} \subset \mathbb{P}^1$; and define the charts (U_j, ϕ_j) where $\phi_j([z_0, z_1]) = \frac{z_{1-j}}{z_j}$. Endow S^2 with the compatible charts you found in problem (2). Show that $\alpha : \mathbb{P}^1 \to S^2$ is holomorphic.