Homework 3

Due: Friday, September 7

1. Since $\mathbb{Z}[2i] \subset \mathbb{Z}[i]$, the "multiplication by one" map induces a map of elliptic curves

$$\begin{array}{ccc} z & \longmapsto & z \\ V_1 \cong \mathbb{C} & \xrightarrow{\widetilde{f}} & V_2 \cong \mathbb{C} \\ \varpi_1 \big\downarrow & & \big\downarrow \varpi_2 \\ \mathbb{C}/\mathbb{Z}[2i] & \xrightarrow{f} & \mathbb{C}/\mathbb{Z}[i] \end{array}$$

   (a) Describe all $P \in V_1$ such that $\varpi_2 \circ \widetilde{f}(P) = \varpi_2(0)$.

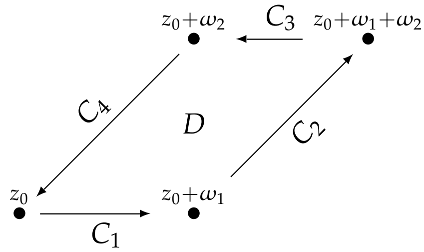
   (b) What is $\ker f$?

   (c) Repeats parts (a) and (b) when $\mathbb{Z}[2i]$ is replaced with $\mathbb{Z}[ni]$ for an arbitrary natural number n.

2. Let h be a meromorphic function, and suppose $z_0 \in \mathbb{C}$, $z_0 \neq 0$. Show that

$$\operatorname{res}_{z_0}\left(\frac{zh'(z)}{h(z)}\right) = \operatorname{ord}_{z_0}(h) \cdot z_0.$$

3. Let $\Lambda \subset \mathbb{C}$ be a lattice, and suppose f is a Λ-periodic function. Let D be a fundamental domain as in (1)



(1)

so that that f has no poles or zeros on $\partial\overline{D}$. Show that:

$$\frac{1}{2\pi i}\int_{\partial\overline{D}} \frac{zf'(z)}{f(z)}dz \in \Lambda. \tag{2}$$

Note that the integrand $zf'(z)/f(z)$ is *not* periodic.

You are welcome to take the following steps. You may want to use the fact that log is actually a multifunction, so that if C is a contour from α to β, and if g is any continuous, nonvanishing function on C, then

$$\int_C \frac{g'(z)}{g(z)}dz = \log(g(z))\Big|_\alpha^\beta \in (\operatorname{Log}(\beta) - \operatorname{Log}(\alpha)) + 2\pi i\mathbb{Z}.$$

(a) Show that $\int_{C_3} \frac{zf'(z)}{f(z)} dz = \int_{-C_1} \frac{zf'(z+\omega_2)}{f(z+\omega_2)} dz + \omega_2 \int_{C_3} \frac{f'(z)}{f(z)} dz$. (HINT: $zf'(z)/f(z) = (z - \omega_2)f'(z)/f(z) + \omega_2 f'(z)/f(z)$.)

(b) Show that $\int_{C_1+C_3} \frac{zf'(z)}{f(z)} dz \in 2\pi i \omega_2 \mathbb{Z}$.

(c) Show (2).

4. Suppose f is meromorphic and Λ-periodic. Let D be a fundamental domain for Λ as in (1), and suppose that f has no zeros or poles on $\partial\overline{D}$.

Suppose the only zeros of f in D are $P_1, \cdots, P_r$, and that f has a zero of order m_i at P_i. Similarly, suppose the only poles of f in D are $Q_1, \cdots, Q_s$, and that f has a pole of order n_j at Q_j. Show that

$$\sum_{i=1}^{r} m_i P_i \equiv \sum_{j=1}^{s} n_j Q_j \bmod \Lambda.$$