Homework 2

Due: Friday, August 31

Throughout this problem set, fix a lattice $\Lambda \subset \mathbb{C}$.

Please review basic facts about contour integrals, residues, etc.

1. The Weierstrass σ-function is

$$\sigma(z) = z \cdot \prod_{\lambda \in \Lambda'} (1 - \frac{z}{\lambda}) \exp(-\frac{z}{\lambda} + (\frac{z}{\lambda})^2/2).$$

(For the purposes of this exercise, infinite products essentially behave like infinite sums; if the terms go to 1 sufficiently rapidly, then convergence is absolute and uniform.) This defines a holomorphic function on all of $\mathbb{C}$, with simple zeros at each $\lambda \in \Lambda$, and nonvanishing elsewhere.

(a) Show that for $z \in \mathbb{C} - \Lambda$,

$$\frac{d^2}{dz^2} \log \sigma(z) = -\wp(z).$$

(b) Suppose $\lambda \in \Lambda$. Show that there are constants $a_\lambda, b_\lambda \in \mathbb{C}$ such that for all $z \in \mathbb{C}$,

$$\sigma(z + \lambda) = \exp(a_\lambda z + b_\lambda)\sigma(z).$$

(HINT: *Given part (a), what can you say about* $\log(\sigma(z + \lambda))$*?*)

2. [cf. Debarre, Remark 2.4] Let $D \subset \mathbb{C}$ be a fundamental domain for Λ which contains the origin.

Fix $z_0 \in D$, $z_0 \neq 0$, and consider the function

$$\psi(z) = \wp(z) - \wp(z_0).$$

(a) Describe the poles of $\psi(z)$.

(b) Show that $\psi(z)$ has exactly two zeros in D, counted with multiplicity.

(c) What are the zeros of ψ? (HINT: $\wp$ *is an even function.*)

3. Suppose that $\alpha \in \mathbb{C}$ satisfies $\alpha\Lambda \subseteq \Lambda$. Show that α is actually an algebraic integer, of degree at most 2. (In other words, show that α satisfies a polynomial of the form $X^2 + pX + q = 0$, with $p, q \in \mathbb{Z}$.) (HINT: *Choose a basis* $\{\omega_1, \omega_2\}$*, and think of* α *as a linear transformation from* Λ *to itself. This means you can write down* α *as a matrix. What can you say about its characteristic polynomial?*)

Extra credit: In the situation of the previous problem, suppose $\alpha \notin \mathbb{Z}$. *Show that* α *is an* imaginary *quadratic integer.*