Homework 12

Due: Friday, November 9

1. *Either do Debarre problem 4.1, or:* Let $V = \mathbb{C}^n$, and let $\Lambda \subset V$ be a lattice.

   (a) Let f be a holomorphic function on (all of) V. Suppose there is a constant d such that for each j, $1 \le j \le n$, we have
   $$\frac{\partial f(z)}{\partial z_j} = d.$$
   Show that f is a polynomial of degree (at most) two. (HINT: *If f is entire, then it has a Taylor expansion valid everywhere.*)

   (b) Suppose that θ is a theta function for Λ which is nowhere vanishing. Show there is a polynomial $P(z)$ of degree at most two such that
   $$\theta(z) = \widetilde{e}(P(z)).$$
   (HINT: *Consider the function* $\log\theta(z)$. *(Why is this well-defined?)*)

2. Suppose σ is in the *lower* half-plane, i.e., $\mathrm{im}(\sigma) < 0$.

   Does the series
   $$\sum_{m\in\mathbb{Z}} \widetilde{e}(\frac{1}{2}(\sigma m^2) + mz)$$
   define a function on $\mathbb{C}$? Explain.

3. *Riemann's theta function really is a theta function* Let $e_1, \cdots, e_g$ be a basis for $V = \mathbb{C}^g$. If $\tau \in \mathfrak{h}_g$, we have defined a lattice $\Lambda_\tau \subset V$; its basis vectors are $\{e_1, \cdots, e_g\}$ and the columns of τ.

   Riemann's theta function is
   $$\vartheta(z) = \vartheta(z,\tau) = \sum_{m\in\mathbb{Z}^g} \widetilde{e}(\frac{1}{2}(m^T\tau m) + m^T z).$$

   Suppose $n \in \mathbb{Z}^g$.

   (a) Show that
   $$\vartheta(z+n) = \vartheta(z).$$

   (b) Show that
   $$\vartheta(z+\tau n) = \widetilde{e}(-\frac{1}{2}n^T\tau n - n^T z)\vartheta(z).$$

   (c) Compare this to the transformation rules for the one-dimensional theta functions.

4. Write a few sentences about the subject of your final project.