Homework 11
Due: Friday, November 2

1. Let V/K be a finite-dimensional vector space, with basis $\{e_1, \cdots, e_n\}$ and dual basis $\ell_1, \cdots, \ell_n$.
   (a) What is $\ell_1 \wedge \cdots \wedge \ell_n(e_1, \cdots, e_n)$?
   (b) Let $\sigma \in S_n$ be a permutation. What is
$$\ell_1 \wedge \cdots \wedge \ell_n(e_{\sigma(1)}, \cdots, e_{\sigma(n)})?$$

2. Consider the projection
$$U := \mathbb{C}^{n+1} - \{0\} \xrightarrow{\alpha} \mathbb{P}^n$$
   (a) Convince yourself (and me) that if η is an r-form on $\mathbb{P}^n$, then $\omega := \alpha^*\eta$ is a $\mathbb{C}^\times$-invariant r-form on U.
   (b) If η is a (p,q)-form, then so is ω. Write
$$\omega = \sum_{j_1<j_2<\cdots<j_p;k_1<\cdots<k_q} \omega_{j_1\cdots j_p;k_1\cdots k_q}(z)dz_{j_1} \wedge \cdots dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}.$$
   There is a function $\beta : \mathbb{C}^\times \to \mathbb{C}^\times$ so that, for all $z \in U$ and all $t \in \mathbb{C}^\times$,
$$\omega_{j_1\cdots j_p;k_1\cdots k_q}(tx) = \beta(tx)\omega_{j_1\cdots j_p;k_1\cdots k_q}(x).$$
   What is β?

   (HINT: *See Debarre section 3.5*)

3. Consider the differential form
$$\omega = \frac{i}{2\pi}\frac{\|z\|^2\left(\sum_{j=0}^n dz_j \wedge d\overline{z}_j\right) - \left(\sum_{j=0}^n \overline{z}_j dz_j\right) \wedge \left(\sum_{k=0}^n z_k d\overline{z}_k\right)}{\|z\|^4}$$
   on $\mathbb{C}^{n+1} - \{0\}$, where $\|z\| = \sqrt{\sum_{j=0}^n z_j\overline{z}_j}$.
   (a) What is the type (p,q) of ω?
   (b) Show that ω is invariant under $\mathbb{C}^\times$.

Extra credit: Let $U = \mathbb{C} - \{0\}$. Show that $H^1_{\mathrm{dR}}(U)$ is not trivial.