Homework 10

Due: Friday, October 26

1. Consider the usual covering $\mathbb{P}^1 = U_0 \cup U_1$; recall that $U_0 = \mathbb{P}^1 - \{[0,1]\} \cong \mathbb{C}$ and $U_1 = \mathbb{P}^1 - \{[1,0]\}$.

   (a) Let z be the coordinate on U_0, and let w be the coordinate on U_1. Describe the function $\phi_1 \circ \phi_0^{-1}$. How are w and z related?

   (b) Suppose that ω is a meromorphic one-form* on $\mathbb{P}^1$ defined by $f(z)dz$ on U_0 and $g(w)dw$ on U_1. Show that
   $$f(z) = g(1/z)(-1/z^2).$$

   (c) Show that there are no non-zero holomorphic one-forms on $\mathbb{P}^1$!.

2. Let $\Lambda \subset V \cong \mathbb{C}$ be a lattice, and let $\pi : V \to V/\Lambda = X$ be the usual projection.

   Suppose $\omega \in \Omega^1(X)$ is a holomorphic one-form on X. Then $\pi^*\omega$ is a periodic holomorphic one-form on V.†

   (a) Show that $\pi^*\omega = adz$ for some $a \in \mathbb{C}$.
   (HINT: *If $\eta \in \Omega^1(V)$, then there is some function $f \in \mathcal{H}(V)$ such that $\eta = f(z)dz$.*)

   (b) What is $\dim_{\mathbb{C}} \Omega^1_X(X)$?

3. Let V be a finite-dimensional vector space over $\mathbb{C}$. A Hermitian form on V is a map
   $$V \times V \xrightarrow{H} \mathbb{C}$$
   such that:

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   $$H(u_1 + u_2, v) = H(u_1, v) + H(u_2, v)$$
   $$H(au_1, v) = aH(u_1, v)$$

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   $$H(u,v) = \overline{H(v,u)}$$

*If z is a coordinate on an open subset $V \subset \mathbb{C}$, a meromorphic one-form on V is an expression $f(z)dz$, where $f(z) \in \mathcal{M}(V)$ is a meromorphic function on V. It turns out that any one-form on $\mathbb{P}^1$ can be written down using the cover $\mathbb{P}^1 = U_0 \cup U_1$; one need not further subdivide either of the U_j's.

†In other words, for each $\lambda \in \Lambda$, if $T_\lambda : V \to V$ is the translation map, then $T_\lambda^*(\pi^*\omega) = \pi^*\omega$.

Suppose H is a Hermitian form on V. Write the real and imaginary parts of H as S and E, so that

$$H(u,v) = S(u,v) + iE(u,v)$$

where $S, E : V \times V \to \mathbb{R}$ are real bilinear.

Show that:

$$H(u, av) = \overline{a}H(u,v) \tag{1}$$
$$H(u,u) \in \mathbb{R} \tag{2}$$
$$S(u,v) = E(iu, v) \tag{3}$$
$$S(iu, iv) = S(u,v) \tag{4}$$
$$E(iu, iv) = E(u,v) \tag{5}$$
$$S(u,v) = S(v,u) \tag{6}$$
$$E(u,v) = -E(v,u) \tag{7}$$

These should all be very brief.

4. It's time to start thinking about a topic for your final project. You can find a (growing, far-from-comprehensive) list of possible topics at

`http://www.math.colostate.edu/~achter/619/help/proj.html`