Homework

Due: Friday, April 10

1. Fix $P_0 \in E(\mathbb{Q})$. Show that there exists some $B = B(P_0, E)$ such that, for all $P \in E(\mathbb{Q})$,
$$h(P_0 + P) \leqslant 2h(P) + B.$$

2. There is a slightly different height function $\hat{h}$ which is harder to define, but easier to work with, than the usual height.

   Recall that given $E/\mathbb{Q}$, there is a constant A such that, for each $P \in E(\mathbb{Q})$,
$$|h(2P) - 4h(P)| \leqslant A.$$

   (a) Show that
$$\left| \frac{1}{4^n} h(2^n P) - h(P) \right| \leqslant A \sum_{1 \leqslant j \leqslant n} \frac{1}{4^j}.$$

   (b) Show that the sequence of numbers
$$\{\frac{1}{4^n} h(2^n P)\}$$
   is Cauchy.

   Then, one defines
$$\hat{h}(P) = \lim_{n \to \infty} \frac{1}{4^n} h(2^n P).$$

3. Show that for each M,
$$\{P \in E(\mathbb{Q}) : \hat{h}(P) < M\}$$
   is finite. (HINT: *Use (a) to show the existence of a* $B = B(E)$ *such that* $\left|h(P) - \hat{h}(P)\right| < B$.)

4. (a) Show that for each $P \in E(\mathbb{Q})$,
$$\hat{h}(2P) = 4\hat{h}(P).$$

   (b) It turns out that, for any $P, Q \in E(\mathbb{Q})$,
$$\hat{h}(P + Q) + \hat{h}(P - Q) = 2\hat{h}(P) + 2\hat{h}(Q)$$
   Use to this to show that for each m,
$$\hat{h}(mP) = m^2 \hat{h}(P).$$

5. (a) Suppose $P \in E(\mathbb{Q})$ is a torsion point. Show that $\hat{h}(P) = 0$.
   (HINT: *The set* $\{2^n P : n \geqslant 1\}$ *is finite.*)

   (b) Suppose $\hat{h}(P) = 0$. Show that P is torsion.
   (HINT: *The set* $\{mP : m \geqslant 1\}$ *has bounded height.*)