Homework 6
Due: Friday, April 3

1. Recall the Bachet curve we studied at the beginning of the semester

$$E : y^2 = x^3 - 2. \tag{1}$$

   (a) For which primes p does E have good reduction at p?

   (b) By explicitly examining all possible x-coordinates of points, compute $\#E(\mathbb{F}_5)$ and $\#E(\mathbb{F}_7)$.

2. Suppose $E/\mathbb{Q}$ is an elliptic curve Let $T \subset= E(\mathbb{Q})_{\text{tors}}$ be the torsion subgroup.

   (a) Show that, for each ℓ,

$$\cup_{n \geqslant 1} E[\ell^n](\mathbb{Q})$$

   is finite. (HINT: *Choose a prime* $p \neq \ell$ *of good reduction for* E.)

   (b) Show that T is finite. (HINT: *Work with two different primes.*)

3. Continue to work with the Bachet curve (1).

   (a) Show that the torsion group of $E(\mathbb{Q})$ is trivial.

   (b) Show that $E(\mathbb{Q})$ is infinite. (HINT: $5^2 = 3^3 - 2$.)

4. Use the

   Isogeny Theorem: Suppose E_1 and E_2 are elliptic curves over $\mathbb{F}_q$. Then the natural map

$$\text{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \longrightarrow \text{Hom}_{\text{Gal}(\mathbb{F}_q)}(T_\ell E_1, T_\ell E_2)$$

   is an isomorphism

   to show that, for a pair of elliptic curves $E_1, E_2/\mathbb{F}_q$, E_1 and E_2 are isogenous if and only if $\#E_1(\mathbb{F}_q) = \#E_2(\mathbb{F}_q)$.

5. Recall that if $X/\mathbb{F}_q$ is a variety, then its zeta function is

$$Z_{X/\mathbb{F}_q}(T) = \exp\Big(\sum_{n \geqslant 1} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n}.$$

   Calculate $Z_{\mathbb{P}^1/\mathbb{F}_p}(T)$.