Homework 5

Due: Friday, February 27

Remember, you are expected to read every question, but you need only be able to answer two of them.

1. Let $E = \mathbb{C}/\Lambda$ be a complex elliptic curve. Explain the following assertions:
   (a) We have
   $$\mathrm{End}(E) \cong \{\alpha \in \mathbb{C} : \alpha\Lambda \subseteq \Lambda\}.$$
   (b) In fact, $\mathrm{End}(E)$ is a torsion-free ring, of rank at most 4 as an $\mathbb{Z}$-module.

2. *Continuation of* (1) Let E be a complex elliptic curve. Show that either $\mathrm{End}(E) \cong \mathbb{Z}$ or $\mathrm{End}(E)$ is isomorphic to an order in a quadratic imaginary field.

3. Let $\alpha : E_1 \to E_2$ be a nonzero homomorphism of elliptic curves. Show that
   (a) $\ker(\alpha)$ is finite.
   (b) In particular, there is some N such that $N \cdot \ker(\alpha) = \mathcal{O}$.

4. (a) Let $\alpha : X \to Y$ be a nonconstant morphism of smooth, projective curves. Sketch briefly why there is a group homomorphism
   $$\mathrm{Pic}^0(Y) \xrightarrow{\alpha} \mathrm{Pic}^0(X)$$
   (b) Let $\alpha \in \mathrm{Hom}(E_1, E_2)$ be nonzero. Use (a) to construct a natural $\hat{\alpha} \in \mathrm{Hom}(E_2, E_1)$.
   (HINT: *There is a canonical isomorphism* of varieties $E_i \xrightarrow{\sim} \mathrm{Pic}^0(E_i)$.)
   It turns out that:

   Theorem If $\deg(\alpha) = N$, then $\hat{\alpha} \circ \alpha = [N]_{E_1}$.

5. "Taking duals" is an anti-involution of $\mathrm{End}(E)$: If $\alpha, \beta \in \mathrm{End}(E)$, then
   $$\widehat{\alpha + \beta} = \widehat{\alpha} + \widehat{\beta}$$
   $$\widehat{\alpha \circ \beta} = \widehat{\beta} \circ \widehat{\alpha}$$

   Show by induction that, for each positive integer m, $\widehat{[m]} = [m]$.
   (The claim is trivial for $m = 1$.)