Homework 4

Due: Friday, February 20

1. Consider an elliptic curve with affine equation
$$y^2 = x^3 + ax^2 + bx + c.$$
   (a) Recall (HW2#2) or look up the duplication formula for a point. Use your formula to algebraically compute
$$\ker([2]) := \{P \in E(\overline{K}) : 2P = \mathcal{O}\}.$$
   (b) Find a formula for $[3]P := P + P + P$.
   (c) Use your formula to calculate
$$\#\{P \in E(\overline{K}) : [3]P = \mathcal{O}\}.$$

2. Let $E = \mathbb{C}/\Lambda$ be a complex elliptic curve, with uniformizing map
$$\mathbb{C} \xrightarrow{\varpi} E = \mathbb{C}/\Lambda$$
   (a) For a natural number N, what is
$$\{z \in \mathbb{C} : N\varpi(z) = \mathcal{O}\}?$$
   (b) Describe the group
$$E[N](\mathbb{C}) = \{\varpi(z) \in \mathbb{C}/\Lambda : N\varpi(z) = \mathcal{O}\}.$$

3. Let $\Lambda \subset \mathbb{C}$ be a lattice. Suppose that $\alpha \in \mathbb{C}$ satisfies $\alpha\Lambda \subseteq \Lambda$.
   (a) Show that α is actually an algebraic integer, of degree at most 2. (In other words, show that α satisfies a polynomial of the form $X^2 + pX + q = 0$, with $p, q \in \mathbb{Z}$.)
   (HINT: *Choose a basis* $\{\omega_1, \omega_2\}$, *and think of* α *as a linear transformation from* Λ *to itself. This means you can write down* α *as a matrix. What can you say about its characteristic polynomial?*)
   (b) Suppose $\alpha \notin \mathbb{Z}$. Show that α is an *imaginary* quadratic integer.

4. *Try this one after we work with the* **Rigidity Lemma** *in class.*
   Let X be an irreducible, projective, group variety. (In other words, X is equipped with a group law
$$X \times X \xrightarrow{\mu} X$$
   which satisifies the usual axioms.)
   Show that this group law is commutative.