Homework 3

Due: Friday, February 13

1. [*Associativity*] Suppose $P, Q, R \in E(K)$; we wish to show that

$$(P+Q)+R = P+(Q+R),$$

where $S+T = (S*T)*\mathcal{O}$, and (thus) $S*T = -(S+T)$. Assume that P, Q and R are sufficiently general, in the sense that

$$\{P, Q, R, P+Q, P+R, Q+R\}$$

are all distinct.

   (a) Show that it suffices to prove that

   $$(P+Q)*R = P*(Q+R).$$

   (b) Consider the lines

   $$\overline{P+Q:R} \text{ and } \overline{P:Q+R}.$$

   Show that it suffices to show

   $$\overline{P+Q:R} \cap \overline{P:Q+R} \subset E.$$

   (c) Consider the following cubic curves, each of which is a union of three lines:

   $$C_1 : \overline{Q:R} \cup \overline{P:Q+R} \cup \overline{\mathcal{O}:P+Q}$$
   $$C_2 : \overline{P+Q:R} \cup \overline{Q+R:\mathcal{O}} \cup \overline{P:Q}$$

   Compute, in terms of P, Q, and R, the nine points of intersection of C_1 and C_2.

   (d) Finish by using the following result:

   Cayley-Bacharach Theorem Let E, C_1 and C_2 be cubic curves, where no two contain a common component; let $C_1 \cap C_2 = \{P_1, \cdots, P_9\}$. Suppose E contains $\{P_1, \cdots, P_8\}$. Then $P_9 \in E$, too.

2. Sketch a proof of the Cayley-Bacharach theorem, by justifying the following claims:

   (a) The set Cubics of all cubic curves is a nine-dimensional space. (HINT: $F(X,Y,Z) = 0$ *and* $\lambda F(X,Y,Z) = 0$ *determine the same curve.*)

   (b) Let $P_1, \cdots, P_8$ be distinct points. Show that

   $$\mathrm{Cubics}(P_1, \cdots, P_8) := \{C \in \mathrm{Cubics} : \text{ each } P_i \in C\}$$

   is one-dimensional.

(c) Let $C_1 = \mathcal{V}_{F_1}$ and $C_2 = \mathcal{V}_{F_2}$. Show that

$$\mathrm{Cubics}(P_1, \cdots, P_8) = \{\mathcal{V}_{\lambda_1 F_1 + \lambda_2 F_2} : \lambda_i \in K\}.$$

(d) Show that

$$\mathrm{Cubics}(P_1, \cdots, P_8) = \mathrm{Cubics}(P_1, \cdots, P_9).$$

3. Given elements $\alpha, \beta \in \mathbb{C}$, the (additive) subgroup they generate is:

$$G_{\alpha,\beta} = \{m\alpha + n\beta : m, n \in \mathbb{Z}\}$$

(a) Give an example where $G_{\alpha,\beta} \cong \mathbb{Z}$.

(b) Give an example where $G_{\alpha,\beta} \cong \mathbb{Z} \oplus \mathbb{Z}$, but $G_{\alpha,\beta}$ is *not* discrete.

4. Suppose $\Lambda \subset \mathbb{C}$ is a lattice, with ordered basis $\{\omega_1, \omega_2\}$. Let D be the (open) parallelogram $\{s\omega_1 + t\omega_2 : 0 < s, t < 1\}$, and let C be a simple, positive closed contour around the boundary of $\overline{D}$. (For example, C is the piecewise-linear contour which visits, successively, 0, ω_1, $\omega_1 + \omega_2$, ω_2, and back to 0.)

Suppose that f is meromorphic, Λ-periodic, and has no poles along C. Show that $\int_C f(z)\,dz = 0$.

REMARK This, combined with the residue theorem, can be used to show that a nonconstant Λ-periodic meromorphic function has at least two poles.