Homework 2
Due: Friday, February 6

You might not be able to do many of these until after class on Monday, February 2.

1. Consider the affine curve
$$E_0 : y^2 = x^3 + ax^2 + bx + c.$$
   (a) Find an equation (i.e., homogeneous polynomial in X, Y and Z) for its projective closure, E.
   (b) Find an equation for E in a neighborhood of $\mathcal{O} := [0, 1, 0]$. (HINT: *Dehomogenize on* Y.)

   Extra : Show that E is smooth at $\mathcal{O}$.

2. We work with the affine curve $C : y^2 - x^3 + x^2$ over $\mathbb{Q}$ and the points $P = (0, 0)$ and $Q = (2, 2)$. Set
$$A = \frac{\mathbb{Q}[x, y]}{y^2 + x^3 - x^2}$$
$$\mathfrak{m}_P = (x, y)A \subset A$$
$$\mathfrak{m}_Q = (x - 2, y - 2)A \subset A.$$
   (a) Graph $C(\mathbb{R})$.
   (b) What is $\dim_{\mathbb{Q}} \mathfrak{m}_P/\mathfrak{m}_P^2$?
   (c) What is $\dim_{\mathbb{Q}} \mathfrak{m}_Q/\mathfrak{m}_Q^2$?

3. Let K be a perfect field and let $\overline{K}$ be an algebraic closure of K. Suppose $P \in \mathbb{P}^2(\overline{K})$ is a point such that, for each $\sigma \in \mathrm{Gal}(\overline{K}/K)$, $P^\sigma = P$. Show that $P \in \mathbb{P}^2(K)$. (HINT: *Here are two different ways to do it: dehomogenize, and use a result about affine spaces; or use Hilbert's theorem 90.*)

4. Let E be the elliptic curve with affine equation
$$y^2 = x^3 + ax^2 + bx + c.$$
   (a) *Without using any formulas*, describe
$$\{P \in E(\overline{K}) : 2P = \mathcal{O}\}.$$
   (HINT: $2P = \mathcal{O}$ *if and only if* $P = -P$.)
   (b) Let $P = (x_0, y_0) \in E(K)$, $P \neq \mathcal{O}$. What is the equation for the tangent line to E at P?
   (c) Find an equation for $2P = P + P$. (HINT: *Intersect the tangent line in (a) with* E.)