Homework 7

Due: ??

- 1. Suppose $\pi: D \to C$ is a Galois cover of smooth projective curves over \mathbb{F}_q , with Galois group G. A point $P \in C$ is called *split* (in this extension) if $\pi^{-1}(P)$ consists of #G points.
 - (a) Suppose *P* is split, and $Q \in \pi^{-1}(P)$. Explain why $\deg(Q) = \deg(P)$.
 - (b) Show that

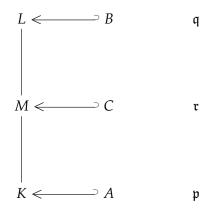
$$\lim_{n\to\infty}\frac{\#\{P\in C:\deg(P)=n,\ P\ \mathrm{split}\}}{\#\{P\in C:\deg(P)=n\}}=\frac{1}{\#G}.$$

(HINT: Remember that we proved (assuming Weil's theorem) that the number of points of degree d on a curve is $\frac{q^d}{d} + O(q^{d/2})$.)

- 2. Just as \mathbb{P}^1 can be realized as the disjoint union of a copy of \mathbb{A}^1 and a point at infinity, for $N \geq 1$ we can decompose \mathbb{P}^N as a disjoint union $\mathbb{P}^N = \mathbb{A}^N \cup \mathbb{P}^{N-1}$.
 - (a) Fix a natural number N, and let \mathbb{F}_q be a finite field. What is

$$\#\mathbb{P}^N(\mathbb{F}_q)$$
?

- (b) Calculate the zeta function $Z_{\mathbb{P}^N/\mathbb{F}_p}(T)$ of \mathbb{P}^N , thought of as a variety over \mathbb{F}_p .
- (c) Given your calculation, what do you think the Betti numbers of $\mathbb{P}^N_{\mathbb{C}}$ are?
- 3. As in class, suppose L/K is a Galois extension; M is an intermediate extension; $A \subset K$ is a Dedekind ring with field of fractions K; B and C the integral closures of A in L and M; $\mathfrak{p} \subset A$ a prime (really, maximal) ideal; $\mathfrak{r} \subset C$ a prime lying over \mathfrak{p} , and $\mathfrak{q} \subset B$ a prime lying over \mathfrak{r} .



- (a) Show that $f(\mathfrak{q}/\mathfrak{p}) = f(\mathfrak{q}/\mathfrak{r})f(\mathfrak{r}/\mathfrak{p})$.
- (b) Show that $e(\mathfrak{q}/\mathfrak{p}) = e(\mathfrak{q}/\mathfrak{r})e(\mathfrak{r}/\mathfrak{p})$
- 4. With notation as in the previous problem, let H = Gal(L/M).

- (a) Show that $D(\mathfrak{q}/\mathfrak{r}) = H \cap D(\mathfrak{q}/\mathfrak{p})$ and $I(\mathfrak{q}/\mathfrak{r}) = H \cap I(\mathfrak{q}/\mathfrak{p})$.
- (b) Suppose H is normal, and let $\rho: G = \operatorname{Gal}(L/K) \to \operatorname{Gal}(M/K) = G/H$ be the quotient map. Show that $\rho(D(\mathfrak{q}/\mathfrak{p})) = D(\mathfrak{r}/\mathfrak{p})$ and $\rho(I(\mathfrak{q}/\mathfrak{p})) = I(\mathfrak{r}/\mathfrak{p})$.