Homework 7

Due: ??

1. Suppose $\pi : D \to C$ is a Galois cover of smooth projective curves over $\mathbb{F}_q$, with Galois group G. A point $P \in C$ is called *split* (in this extension) if $\pi^{-1}(P)$ consists of $\#G$ points.

   (a) Suppose P is split, and $Q \in \pi^{-1}(P)$. Explain why $\deg(Q) = \deg(P)$.

   (b) Show that
   $$\lim_{n\to\infty} \frac{\#\{P \in C : \deg(P) = n,\ P \text{ split}\}}{\#\{P \in C : \deg(P) = n\}} = \frac{1}{\#G}.$$
   (HINT: *Remember that we proved (assuming Weil's theorem) that the number of points of degree d on a curve is* $\frac{q^d}{d} + O(q^{d/2})$.)

2. Just as $\mathbb{P}^1$ can be realized as the disjoint union of a copy of $\mathbb{A}^1$ and a point at infinity, for $N \geq 1$ we can decompose $\mathbb{P}^N$ as a disjoint union $\mathbb{P}^N = \mathbb{A}^N \cup \mathbb{P}^{N-1}$.

   (a) Fix a natural number N, and let $\mathbb{F}_q$ be a finite field. What is
   $$\#\mathbb{P}^N(\mathbb{F}_q)?$$

   (b) Calculate the zeta function $Z_{\mathbb{P}^N/\mathbb{F}_p}(T)$ of $\mathbb{P}^N$, thought of as a variety over $\mathbb{F}_p$.

   (c) Given your calculation, what do you think the Betti numbers of $\mathbb{P}^N_{\mathbb{C}}$ are?

3. As in class, suppose L/K is a Galois extension; M is an intermediate extension; $A \subset K$ is a Dedekind ring with field of fractions K; B and C the integral closures of A in L and M; $\mathfrak{p} \subset A$ a prime (really, maximal) ideal; $\mathfrak{r} \subset C$ a prime lying over $\mathfrak{p}$, and $\mathfrak{q} \subset B$ a prime lying over $\mathfrak{r}$.

$$\begin{array}{ccccc} L & \hookleftarrow & B & & \mathfrak{q} \\ | & & & & \\ M & \hookleftarrow & C & & \mathfrak{r} \\ | & & & & \\ K & \hookleftarrow & A & & \mathfrak{p} \end{array}$$

   (a) Show that $f(\mathfrak{q}/\mathfrak{p}) = f(\mathfrak{q}/\mathfrak{r})f(\mathfrak{r}/\mathfrak{p})$.

   (b) Show that $e(\mathfrak{q}/\mathfrak{p}) = e(\mathfrak{q}/\mathfrak{r})e(\mathfrak{r}/\mathfrak{p})$

4. With notation as in the previous problem, let $H = \mathrm{Gal}(L/M)$.

(a) Show that $D(\mathfrak{q}/\mathfrak{r}) = H \cap D(\mathfrak{q}/\mathfrak{p})$ and $I(\mathfrak{q}/\mathfrak{r}) = H \cap I(\mathfrak{q}/\mathfrak{p})$.

(b) Suppose H is normal, and let $\rho : G = \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K) = G/H$ be the quotient map. Show that $\rho(D(\mathfrak{q}/\mathfrak{p})) = D(\mathfrak{r}/\mathfrak{p})$ and $\rho(I(\mathfrak{q}/\mathfrak{p})) = I(\mathfrak{r}/\mathfrak{p})$.