
Homework 6

Due: Monday, April 2

1. The Riemann-Roch theorem, as stated in class, says the following:

   Let C/k be a smooth projective curve. There exists an integer $g \geq 0$ and a divisor ω on C such that, for each $D \in \mathrm{Div}(C)$,

   $$\ell(D) = \deg(D) + 1 - g + \ell(\omega - D).$$

   Recall also that a nonconstant rational function on C must have at least one pole, and thus

   $$\mathcal{L}(0) = k$$
   $$\ell(0) = \dim \mathcal{L}(0) = 1.$$

   Show the following consequences of Riemann-Roch:

   (a) $\ell(\omega) = g$.

   (b) $\deg(\omega) = 2g - 2$.

   (c) If $\deg D \geq 2g - 1$, then $\ell(D) = \deg(D) - g + 1$.

2. Let $(\infty) \in \mathbb{P}^1_k$ be the point at infinity, so that $\mathbb{P}^1_k = \mathbb{A}^1_k \cup \{(\infty)\}$. Recall that the function field of $\mathbb{P}^1_k$ is $k(\mathbb{P}^1) \cong k(T)$.

   (a) Suppose $n \geq 0$. Show that

   $$\mathcal{L}(n \cdot (\infty)) = \{f \in K[T] : \deg f \leq n\}.$$

   (b) What is $\ell(n \cdot (\infty))$?

   (c) Use this and Riemann-Roch to show that the genus of $\mathbb{P}^1$ is zero.

3. Let C/k be a smooth projective curve, and suppose $P \in C$ is a point of degree one.

   (a) Calculate $\ell(nP)$ for all n.

   (b) Show there is no function $f \in k(C)$ with a pole of order one at P and no other poles.

   (c) Let $\{1, x\}$ be a basis for $\mathcal{L}(2P)$. Why must $(x)_\infty = 2P$?

   (d) Let $\{1, x, y\}$ be a basis for $\mathcal{L}(3P)$. Why must $(y)_\infty = 3P$?

   (e) Find a basis for $\mathcal{L}(4P)$ in terms of products of 1, x and y. (HINT: *If $f \in \mathcal{L}(D)$ and $g \in \mathcal{L}(E)$, then $fg \in \mathcal{L}(D + E)$; why?*)

   (f) Do the same for $\mathcal{L}(5P)$.

   (g) What happens with $\mathcal{L}(6P)$?

4. Let $C/\mathbb{F}_q$ be a smooth projective curve. In class, we will define the zeta function of C (or its function field K) via

$$\zeta_K(s) = \sum_{n\geq 1} b_n q^{-ns}$$

or, using the variable $u = q^{-s}$,

$$Z_K(s) = \sum_{n\geq 1} b_n u^n$$

where

$$b_n = \#\{D \in \operatorname{Div}(C) : D \geq 0, \deg(D) = n\}.$$

If $X/\mathbb{F}_q$ is any variety over $\mathbb{F}_q$, let $N_X(r) = \#X(\mathbb{F}_{q^r})$. One defines its zeta function by

$$\widetilde{Z}_X(u) = \exp(\sum_{r\geq 1} N_X(r)\frac{u^r}{r}).$$

Show that these definitions are the same, i.e., that

$$Z_C(u) = \widetilde{Z}_C(u).$$

5. Using which ever definition you prefer, show that: $Z_{\mathbb{P}^1/\mathbb{F}_q}(u) = \frac{1}{1-u}1 - qu$.

6. What is the relation between $Z_{\mathbb{P}^1/\mathbb{F}_q}(u)$ and $Z_{\mathbb{A}^1/\mathbb{F}_q}(u)$? (Equivalently, what is the relation between $\zeta_{\mathbb{F}_q(T)}(s)$ and $\zeta_{\mathbb{F}_q[T]}(s)$?) Explain.