Homework 5
Due: Friday, February 24

For simplicity, let k be an algebraically closed field in which 6 is invertible, such as $k = \mathbb{C}$.

1. Let $C = k[x,y]/(y^2 - x^3 - x)$. Let $\mathfrak{m} = (x,y)$.
   (a) Show that $\mathfrak{m}$ is maximal. (HINT: *What can you say about the quotient ring $A/\mathfrak{m}$?*)
   (b) Show that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
2. Consider the ring $D = \mathbb{Z}[\sqrt{5}]$.
   (a) Show that D is not integrally closed.
   (b) Find a maximal ideal $\mathfrak{m} \subset D$ such that the localization $D_\mathfrak{m}$ is not a discrete valuation ring.

The polynomial

$$f(x,y) = y^2 - x^3$$

is irreducible, and thus the ring

$$A = \frac{k[x,y]}{f(x,y)}$$

is an integral domain, say with field of fractions $K = \operatorname{Frac} A$.

3. Show that A is not integrally closed:
   (a) Show that $\frac{y}{x} \in K$ is integral over A.
   (b) Show that $\frac{y}{x} \notin A$.
4. Let $\mathfrak{m} \subseteq A$ be the ideal $\mathfrak{m} = (x,y)$.
   (a) Show that $\mathfrak{m}$ is maximal.
   (b) Show that $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2$.
5. Let $B = A[\frac{y}{x}]$; we have $A \subset B \subset K$. Show that in fact $B = k[\frac{y}{x}]$.(HINT: *In other words, think about the subring of* $\operatorname{Frac} \frac{k[x,y]}{(y^2-x^3-x)}$ *generated by the element $\frac{y}{x}$. Show that it contains x and y, too.)*
   In fact, $B \cong k[\frac{y}{x}]$, and the element $\frac{y}{x}$ is transcendental over k. Therefore, B is integrally closed, and in fact it is the integral closure of A.

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