Homework 4

Due: Friday, February 17

1. Define a function δ on the set of (monic) polynomials in $\mathbb{F}_q[T]$:

$$\delta(f(T)) = \begin{cases} 1 & f(T) \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$$

   and the associated Dirichlet series

$$B(s) = \sum_{f(T)} \delta(f) \, \|f\|^{-s}$$

   (a) Is δ strongly multiplicative? Weakly multiplicative?

   (b) Express $B(s)$ as an Euler product. In other words, find an explicit expression $B_p(s)$ so that

$$B(s) = \prod_{P(T)} B_p(s).$$

   (HINT: *$B_p(s)$ is an easy polynomial in $\|P\|^{-s}$.*)

2. (a) Show that

$$B(s) = \frac{\zeta_{\mathbb{A}}(s)}{\zeta_{\mathbb{A}}(2s)}.$$

   (HINT: $\frac{1-z^2}{1-z} = 1 + z$.)

   (b) Give an explicit formula for b_n, the number of squarefree polynomials of degree n. (HINT: *Note that $B(s) = \sum_{n \geq 0} b_n q^{-ns}$.*)

3. Fix a polynomial $f(T) \in \mathbb{F}_q[T]$. Let

$$\begin{aligned} S_f &= \{g(T) \in \mathbb{F}_q[T] : g(T) | f(T)\} \\ A_d &= \{g(T) \in \mathbb{F}_q[T] : \deg g = d\} \\ S_{d,f} &= S_f \cap A_d \end{aligned}$$

   Show that if $d \geq \deg(f)$, then

$$\frac{\# S_{d,f}}{\# A_d} = \|f(T)\|^{-1}.$$

4. (a) Let $P_1, \cdots, P_t \in \mathbb{F}_q[T]$ be distinct irreducible polynomials. Give a heuristic argument showing that the proportion of polynomials $f(T) \in \mathbb{F}_q[T]$ which are not divisible by any $P_i(T)^2$ is

$$\prod_{i=1}^{t} (1 - \|P_i(T)\|^2)^{-1}.$$

   (b) Give a heuristic argument to show that the proportion of square-free polynomials in $\mathbb{F}_q[T]$ is $\zeta_{\mathbb{A}}(2)^{-1}$.