9 Zeta function of a variety

Let X be a variety over $\mathbb{F}_q$, i.e., a reduced scheme of finite type.

For an extension field $\mathbb{F}_{q^m}$,

$$X(\mathbb{F}_{q^m}) = X(\operatorname{Spec}\mathbb{F}_{q^m}) = \operatorname{Mor}_{\mathbb{F}_q}(\operatorname{Spec}\mathbb{F}_{q^m}, X)$$

is the set of $\mathbb{F}_q$-valued points of X.

For each natural number m, set

$$N_m(X) = \#X(\mathbb{F}_{q^m}).$$

Remark $N_m(X)$ is finite. Indeed, X is a finite union of open affine schemes, so it suffices to show this for X affine. Since $X = \operatorname{Spec} A$ with A finitely generated, $A \cong \mathbb{F}_q[T_1, \cdots, T_n]/I$ for some ideal. So $X(\mathbb{F}_{q^m}) \subseteq \mathbb{A}^n(\mathbb{F}_{q^m})$; but the latter set is clearly finite.

Question 9.1. *Fix a base field $\mathbb{F}_q$.*

a. Calculate $N_m(\mathbb{A}^n_{\mathbb{F}_q})$.

b. Calculate $N_m(\mathbb{P}^n_{\mathbb{F}_q})$.

Let X_{cl} be the set of closed points of X (as a scheme).

Question 9.2. *Let $X = \operatorname{Spec}\mathbb{F}_{q^r}$, thought of as a scheme over $\mathbb{F}_q$. What is $\#X(\mathbb{F}_{q^m})$? (*HINT: The answer depends on whether $r|m$.*)*

If $P \in X_{\mathrm{cl}}$, the residue field at P is $\kappa(P)$; the degree of P is $\deg(P) = [\kappa(P) : \mathbb{F}_q]$. Let

$$X_{\mathrm{cl}}^{(d)} = \{P \in X_{\mathrm{cl}} : \deg(P) = d\}.$$

Question 9.3. *Give a formula for $N_m(X)$ in terms of $\#X_{\mathrm{cl}}^{(d)}$. (*HINT: Typically, several different d show up for a given m.*)*

The Hasse-Weil zeta function of $X/\mathbb{F}_q$ is

$$Z(X/\mathbb{F}_q, T) = \exp\left(\sum_{m\geq 1} \frac{N_m(X)}{m} T^m\right) \in \mathbb{Q}[\![T]\!].$$

Question 9.4. *Suppose X and Y are disjoint varieties over $\mathbb{F}_q$. Show that*

$$Z(X \cup Y/\mathbb{F}_q, T) = Z(X/\mathbb{F}_q, T) \cdot Z(Y/\mathbb{F}_q, T).$$

Question 9.5. *a. Compute* $Z(\mathbb{A}^n/\mathbb{F}_q, T)$.

b. Compute $Z(\mathbb{P}^n/\mathbb{F}_q, T)$.

Question 9.6. *Show that*

$$Z(X/\mathbb{F}_q, T) = \prod_{P \in X_{\mathrm{cl}}} (1 - T^{\deg(P)})^{-1}.$$

*(*HINT: It's a little easier to analyze $\log Z(X/\mathbb{F}_q, T)$.*)*