Name:__________________________

1	2	3	4	total

Midterm

Each problem is worth twelve points, evenly divided among subproblems. No notes or books allowed.

1. Consider the following three recursions:

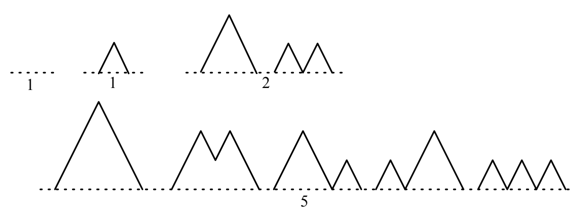
$$\begin{aligned} A(n,k) &= A(n-1,k-1) + kA(n-1,k-1) \\ B(n,k) &= B(n,k-1) + B(n-1,k) \\ C(n,k) &= C(n-1,k-1) + C(n-1,k) \end{aligned}$$

   and the following three counting functions:

   $\binom{n}{k}$ subsets of $[n]$ of size k.
   $M(n,k)$ multisets of $[n]$ of size k;
   $S(n,k)$ partitions of $[n]$ into k nonempty blocks.

   Which recurrence is satisfied by which function? Justify.

2. (a) Show that the number of different mountain ranges you can draw with n upstrokes and n downstrokes is given by the Catalan number C_{n+1}:



   (b) A clown stands at the edge of a swimming pool with a bowl with n red and n blue balls. He randomly draws balls from it. If he draws a red ball he takes a step back. If he draws a blue ball he takes a step forward. Assuming that the steps are always the same length, what is the probability that the clown stays dry?

3. Give a combinatorial argument to show that

$$\sum_{j=1}^{n} j^2 = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

   (HINT: *Consider the set* $\mathcal{S} = \{(i,j,k) : 0 \le i, j < k \le n\}$.)

4. Use generating functions to show that

$$\sum_{j=1}^{n} j^2 = M(4, n-1) + M(4, n-2).$$