Homework 9
Due: Wednesday, April 8

Throughout, let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$, and suppose $T \in \mathcal{L}(V)$.

1. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator whose matrix, in the standard basis, is

$$[T]_{\mathcal{E}} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

Verify that the minimal polynomial of $T$ is $(x - 1)(x - 2)$.

2. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator with matrix

$$[T]_{\mathcal{E}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

Find the minimal polynomials for the action of $T$ on each of the following vectors:

(a) $u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(b) $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(c) $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

3. Suppose $v, w \in V$ and $z \in \text{span}(v, w)$.

(a) Show that $\mu_{T,z}(x)$ divides $\mu_{T,v}(x) \cdot \mu_{T,w}(x)$.

(b) Prove a little more: $\mu_{T,z}(x)$ divides $\text{lcm}(\mu_{T,v}(x), \mu_{T,w}(x))$.

4. In the situation of problem 3, suppose that $\mu_{T,v}(x)$ and $\mu_{T,w}(x)$ are relatively prime, and that $z = av + bw$ with $a, b \neq 0$.

(a) Show that $\mu_{T,z}(x) = \mu_{T,v}(x) \cdot \mu_{T,w}(x)$. (HINT: If $P(x), Q(x) \in \mathbb{F}[x]$, then $P$ and $Q$ are relatively prime $\iff$ lcm$(P(x), Q(x)) = P(x) \cdot Q(x)$.)

(b) Show that $\dim \text{span}(T, z) = \dim \text{span}(T, v) + \dim \text{span}(T, w)$. (HINT: For a vector $u \in V$, how can you tell $\dim \text{span}(T, u)$ by looking at $\mu_{T,u}(x)$?)