Homework 9

Due: Wednesday, April 8

Throughout, let V be a finite-dimensional vector space over a field $\mathbb{F}$, and suppose $T \in \mathcal{L}(V)$.

1. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator whose matrix, in the standard basis, is
$$[T]_{\mathcal{E}} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$
Verify that the minimal polynomial of T is $(x-1)(x-2)$.

2. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator with matrix
$$[T]_{\mathcal{E}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}.$$
Find the minimal polynomials for the action of T on each of the following vectors:
   (a) $u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
   (b) $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
   (c) $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

3. Suppose $v, w \in V$ and $z \in \operatorname{span}(v, w)$.
   (a) Show that $\mu_{T,z}(x)$ divides
$$\mu_{T,v}(x) \cdot \mu_{T,w}(x).$$
   (b) Prove a little more: $\mu_{T,z}(x)$ divides $\operatorname{lcm}(\mu_{T,v}(x), \mu_{T,w}(x))$.

4. In the situation of problem 3, suppose that $\mu_{T,v}(x)$ and $\mu_{T,w}(x)$ are relatively prime, and that $z = av + bw$ with $a, b \neq 0$.
   (a) Show that $\mu_{T,z}(x) = \mu_{T,v}(x) \cdot \mu_{T,w}(x)$. (HINT: *If $P(x), Q(x) \in \mathbb{F}[x]$, then P and Q are relatively prime* $\iff \operatorname{lcm}(P(x), Q(x)) = P(x) \cdot Q(x)$.)
   (b) Show that $\dim \operatorname{span}(T, z) = \dim \operatorname{span}(T, v) + \dim \operatorname{span}(T, w)$. (HINT: *For a vector $u \in V$, how can you tell* $\dim \operatorname{span}(T, u)$ *by looking at* $\mu_{T,u}(x)$*?*)