Homework 8

Due: Wednesday, April 1

As always, V is a vector space over a field $\mathbb{F}$.

1. Suppose $T \in \mathcal{L}(V)$ and $v_1, \cdots, v_r \in V$. Suppose that for each $1 \leq i \leq r$, $T(v_i) \in \mathrm{span}(v_1, \cdots, v_r)$. Show that $\mathrm{span}(v_1, \cdots, v_r)$ is T-invariant. *The converse is very easy to show.*
2. Suppose $T \in \mathcal{L}(V)$.
   (a) Show $\ker(T)$ is T-invariant.
   (b) Show $\mathrm{im}(T)$ is T-invariant.
3. Suppose $\dim V = n$ and $T \in \mathcal{L}(V)$. Use the result of [KK] ex. 5.1.2 to show that T has at most n distinct eigenvalues. *Do not use the characteristic polynomial!*
4. Suppose $\dim V = n$ and $T \in \mathcal{L}(V)$ has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$.
   (a) Explain (briefly) why the characteristic polynomial of T is
   $$\chi_T(X) = (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_n).$$
   (b) Suppose v_i is an eigenvector for λ_i. Show that
   $$\chi_T(T)(v_i) = 0.$$
   (c) Show that $\chi_T(T) = 0$, i.e., that for any $v \in V$, $\chi_T(T)(v) = 0$.

   Compare also [KK] ex. 5.1.3.