Homework 7

Due: Wednesday, March 11

As always, V is a vector space over the field $\mathbb{F}$.

1. Let ψ be a bilinear form on V.[1] Define two new functions on $V \times V$ by

$$\psi_s(v_1, v_2) = \frac{1}{2}(\psi(v_1, v_2) + \psi(v_2, v_1))$$
$$\psi_a(v_1, v_2) = \frac{1}{2}(\psi(v_1, v_2) - \psi(v_2, v_1)).$$

   (a) Show that ψ_s is a symmetric bilinear form on V.

   (b) Show that ψ_a is an alternating bilinear form on V.

   (c) Show that $\psi = \psi_s + \psi_a$, i.e., that for any $v_1, v_2 \in V$,

$$\psi(v_1, v_2) = \psi_s(v_1, v_2) + \psi_a(v_1, v_2).$$

2. Recall that S_3 is the group of permutations on $\{1, 2, 3\}$. For each of the six elements $\sigma \in S_3$, compute $\mathrm{sgn}(\sigma)$ in two different ways:

   (a) Express σ as a product of t transpositions, and compute $(-1)^t$;

   (b) Compute $|\sigma|$, the number of orbits of the action of σ on $\{1, 2, 3\}$, and compute $(-1)^{3-|\sigma|}$.

3. Suppose ϕ is an alternating trilinear form on V. Suppose $v_1, v_2, v_3 \in V$ and $a_{ij} \in \mathbb{F}$ for $1 \leq i, j \leq 3$.

   Explicitly compute

$$\phi(\sum_{i=1}^{3} a_{i1} v_i, \sum_{i=1}^{3} a_{i2} v_i, \sum_{i=1}^{3} a_{i3} v_i).$$

4. Suppose $\dim V = n$, and $\{v_1, \cdots, v_n\}$ is a basis. In class, we defined a nontrivial n-linear alternating form D in the following way: If $u_1, \cdots, u_n \in V$ with

$$u_j = \sum_{i=1}^{n} a_{ij} v_i,$$

   then

$$D(u_1, \cdots, u_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(j),j}.$$

[1] Also, suppose the characteristic of k is not two.

Define vectors $u'_1, \cdots, u'_n$ by

$$u'_i = \sum_{j=1}^{n} a_{ij} v_j.$$

(If you prefer, think of $u'_j = \sum_{i=1}^{n} b_{ij} v_i$, where $b_{ij} = a_{ji}$.)

Show that

$$D(u'_1, \cdots, u'_n) = D(u_1, \cdots, u_n).$$

This shows that if $A \in \mathrm{Mat}_n(\mathbb{F})$ *is a matrix with transpose* A^{tr}*, then* $\det(A) = \det(A^{\mathrm{tr}})$.