As always, $V$ is a vector space over the field $F$.

1. Let $\psi$ be a bilinear form on $V$.1 Define two new functions on $V \times V$ by

$$\psi_s(v_1, v_2) = \frac{1}{2} (\psi(v_1, v_2) + \psi(v_2, v_1))$$

$$\psi_a(v_1, v_2) = \frac{1}{2} (\psi(v_1, v_2) - \psi(v_2, v_1)).$$

(a) Show that $\psi_s$ is a symmetric bilinear form on $V$.
(b) Show that $\psi_a$ is an alternating bilinear form on $V$.
(c) Show that $\psi = \psi_s + \psi_a$, i.e., that for any $v_1, v_2 \in V$,

$$\psi(v_1, v_2) = \psi_s(v_1, v_2) + \psi_a(v_1, v_2).$$

2. Recall that $S_3$ is the group of permutations on $\{1, 2, 3\}$. For each of the six elements $\sigma \in S_3$, compute $\text{sgn}(\sigma)$ in two different ways:

(a) Express $\sigma$ as a product of $t$ transpositions, and compute $(-1)^t$;
(b) Compute $|\sigma|$, the number of orbits of the action of $\sigma$ on $\{1, 2, 3\}$, and compute $(-1)^{3-|\sigma|}$.

3. Suppose $\phi$ is an alternating trilinear form on $V$. Suppose $v_1, v_2, v_3 \in V$ and $a_{ij} \in F$ for $1 \leq i, j \leq 3$.

Explicitly compute

$$\phi(\sum_{i=1}^{3} a_{i1} v_i, \sum_{i=1}^{3} a_{i2} v_i, \sum_{i=1}^{3} a_{i3} v_i).$$

4. Suppose $\dim V = n$, and $\{v_1, \ldots, v_n\}$ is a basis. In class, we defined a nontrivial $n$-linear alternating form $D$ in the following way: If $u_1, \ldots, u_n \in V$ with

$$u_j = \sum_{i=1}^{n} a_{ij} v_i,$$

then

$$D(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(j), j}.$$

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1 Also, suppose the characteristic of $k$ is not two.
Define vectors $u'_1, \ldots, u'_n$ by

$$u'_i = \sum_{j=1}^{n} a_{ij} v_j.$$  

(If you prefer, think of $u'_j = \sum_{i=1}^{n} b_{ij} v_i$, where $b_{ij} = a_{ji}$.)

Show that

$$D(u'_1, \ldots, u'_n) = D(u_1, \ldots, u_n).$$

This shows that if $A \in \text{Mat}_n(\mathbb{F})$ is a matrix with transpose $A^\text{tr}$, then $\det(A) = \det(A^\text{tr})$.  

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