Homework 5

Due: Wednesday, February 25

Throughout, let W and V be vector spaces over a field $\mathbb{F}$.

1. Consider the following matrix $A \in \mathrm{Mat}_{5,5}(\mathbb{Q})$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 5 & -3 & 2 & 11 & -1 \\ 9 & 2 & 1 & 32 & 13 \end{pmatrix};$$

   it represents a linear transformation from $\mathbb{Q}^5$ to $\mathbb{Q}^5$ (with its standard basis). What is the kernel of this transformation? What is it image? (See also exercise [KK]2.5.12, in which the same problem is phrased in terms of rows, instead of columns.)

2. Suppose $T \in \mathcal{L}(W, V)$ is actually an isomorphism. Suppose that $\{w_1, \cdots, w_n\}$ is a basis for W. Prove that $\{T(w_1), \cdots, T(w_n)\}$ is a basis for V, in two steps:
   (a) Show that $\{T(w_1), \cdots, T(w_n)\}$ spans V. (HINT: *T is surjective.*)
   (b) Show that $\{T(w_1), \cdots, T(w_n)\}$ is linearly independent. (HINT: *T is injective.*)

3. Let W and V be vector spaces over $\mathbb{F}$. Suppose that W and V are isomorphic, and that $\dim(W) = n$. Show that $\dim(V) = n$. (HINT: *Use problem 2.*)

4. Suppose $T \in \mathcal{L}(W, V)$. Recall that

$$\ker(T) = \{w \in W : T(w) = 0\}.$$

   Prove that $\ker(T)$ is a subspace of W.

5. Suppose $T \in \mathcal{L}(W, V)$, with W and V finite-dimensional.
   (a) Suppose $\dim W > \dim V$. Prove that T is not injective.
   (b) Suppose $\dim W < \dim V$. Prove that T is not surjective.
   (c) Is the converse of (a) true? If T is not injective, does it follow that $\dim W < \dim V$? Explain. What about (b)?

Professor Jeff Achter
Colorado State University