Homework 4

Due: Wednesday, February 18

Remember: If $\mathcal{B} = \{v_1, \cdots, v_n\}$ is a basis for V, and if $v \in V$, then there are unique $a_1(v), \cdots, a_n(v) \in \mathbb{F}$ such that $v = \sum a_i(v) v_i$; and the associated column vector is

$$[v]_{\mathcal{B}} = \begin{pmatrix} a_1(v) \\ a_2(v) \\ \vdots \\ a_n(v) \end{pmatrix}.$$

Similarly, if $T \in \mathcal{L}(W, V)$, if $\mathcal{C} = \{w_1, \cdots, w_n\}$ is a basis for W and if $\mathcal{B}$ is a basis for V, then we set

$$[T]_{\mathcal{B} \leftarrow \mathcal{C}} = ([T(w_1)]_{\mathcal{B}}\ [T(w_2)]_{\mathcal{B}} \cdots [T(w_n)]_{\mathcal{B}}) \in \mathrm{Mat}_{m,n}(\mathbb{F}).$$

These definitions are engineered so that matrix multiplication computes the effect of a linear transformation:

$$[T(w)]_{\mathcal{B}} = [T]_{\mathcal{B} \leftarrow \mathcal{C}} \cdot [w]_{\mathcal{C}}.$$

1. Consider the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 14 \\ 22 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 9 \\ 14 \end{pmatrix}$$

   Let $\mathcal{B} = (u_1, u_2)$ and $\mathcal{C} = (v_1, v_2)$; each one is a basis for $\mathbb{R}^2$.

   (a) Express u_1 and u_2 in terms of the basis $\mathcal{C}$, and do the same for v_1 and v_2 in terms of $\mathcal{B}$. That is, compute:

$$[u_1]_{\mathcal{C}}, [u_2]_{\mathcal{C}}, [v_1]_{\mathcal{B}}, [v_2]_{\mathcal{B}}.$$

   (b) What is $[\mathrm{id}]_{\mathcal{C} \leftarrow \mathcal{B}}$?

2. Recall that $\mathbb{R}[z]_3$ is the vector space of polynomials of degree at most 3.

   Let $\mathcal{B} = \{1, z, z^2, z^3\}$; let $\mathcal{C} = \{1, 2z, 3z^2, 4z^3\}$. Each is a basis of $\mathcal{P}_3$.

   Let $D : \mathcal{P}_3 \to \mathcal{P}_3$ be the function $f(z) \mapsto f'(z)$.

   (a) Show that D is a linear transformation.

   (b) What is $[D]_{\mathcal{B} \leftarrow \mathcal{B}}$?

   (c) What is $[D]_{\mathcal{C} \leftarrow \mathcal{B}}$?

3. Suppose that W, V and U are vector spaces over $\mathbb{F}$, that $f \in \mathcal{L}(V, U)$, and that $g \in \mathcal{L}(W, V)$. Show that $f \circ g \in \mathcal{L}(W, U)$, i.e., that $f \circ g$ is a linear transformation from W to U.

4. Let W and V be vector spaces over $\mathbb{F}$; let $\mathcal{C}$ be a basis for W, and let $\mathcal{B}$ be a basis for V. Suppose that $S, T \in \mathcal{L}(W, V)$. Show that

$$[S+T]_{\mathcal{B}\leftarrow\mathcal{C}} = [S]_{\mathcal{B}\leftarrow\mathcal{C}} + [T]_{\mathcal{B}\leftarrow\mathcal{C}}.$$

5. Let $v_1, \cdots, v_n$ be a basis for V. We showed that if we write $v = \sum a_i(v)v_i$, then $a_i(v+w) = a_i(v) + a_i(w)$ and $a_i(bv) = b \cdot a_i(v)$. In other words, the map

$$V \xrightarrow{v_i^*} \mathbb{F}$$

$$v \longmapsto a_i(v)$$

is a linear transformation, which we denote $v_i^* \in \mathcal{L}(V, \mathbb{F})$. The purpose of this problem is to show that $\{v_1^*, \cdots, v_n^*\}$ is a basis for $\mathcal{L}(V, \mathbb{F})$.

   (a) Show that

$$v_i^*(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

   (b) What is $(\sum_i b_i v_i^*)(v_j)$?

   (c) Show that the set $\{v_1^*, \cdots, v_n^*\} \subset \mathcal{L}(V, \mathbb{F})$ is linearly independent. (HINT: *Suppose that $\sum b_i v_i^*$ is the zero linear transformation. Evaluate at a vector v_j. What must b_j be?*)

   (d) Show that the set $\{v_1^*, \cdots, v_n^*\} \subset \mathcal{L}(V, \mathbb{F})$ spans. (HINT: *Given $T \in \mathcal{L}(V, \mathbb{F})$, it suffices to find $b_1, \cdots, b_n \in \mathbb{F}$ such that for each j, $T(v_j) = (\sum b_i v_i^*)(v_i)$. Why?*)